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Classes of Extensions and Resolutions

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CLASSES OF EXTENSIONS AND RESOLUTIONS

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A class of resolutions of objects of an abelian category determines a theory of derived functors if each morphism between objects extends to a morphism, unique to within homotopies, between their resolutions. This paper is primarily concerned with resolutions canonically associated with certain natural classes of extensions (E-functors), and the known examples are constructed by using pairs of adjoint functors. An inclusion between two E-functors on the same category induces natural transformations between functors derived from their associated resolutions, and other relations exist in the form of invariant exact couples. The relations simplify for the special and frequently occurring class of 'central' inclusions of E-functors; in particular the operations of forming satellites of a functor on the two resolutions commute. Amongst various applications the general theory provides generalizations of: results on groups of extensions of modules over Dedekind domains; the Hochschild–Serre spectral sequences in the homology theory of groups; the spectral sequences for coherent algebraic sheaves that determine Ext by means of vector bundle resolutions and affine coverings.

Introduction

The ideas of relative homological algebra have been formulated for categories of modules by Hochschild (1956), and for abstract categories by Heller (1958) and Buchsbaum (1959). The common feature of these papers is the selection of a class of extensions or, equivalently, a class of monomorphisms and epimorphisms. In Hochschild's paper it is the class of extensions which split over a given subring of the ring of operators (we always assume that a ring has an indentity and its subrings contain this identity). The class of extensions is used either to construct resolutions of objects of the category, and so obtain the values of derived functors as homology objects (functors are assumed to be additive unless it is otherwise stated), or to construct the relative derived functors of Hom as equivalence classes of multiple extensions. In this paper we study only relative homological algebras on an abelian category. Our primary concern is with the relations between the derived functors constructed from two classes of extensions one of which contains the other, and the construction of relative homological algebras on an abelian category by means of ideals in its ring of endomorphisms, or pairs of adjoint functors—not necessarily additive.

In § 1, so that the conditions to be satisfied by a class of extensions can be conveniently expressed, we define E-functors. They are functors of two variables whose values are groups of extensions, in particular Ext^1 is an E-functor, and they have a simple relationship with the h.f. classes of morphisms defined by Buchsbaum (1959). The theory of h.f. classes shows that an E-functor Θ determines a graded ringoid with multiplication the Yoneda product; the elements of this ringoid are equivalence classes of multiple extensions, the component of degree zero is Hom, and the component of degree one is Θ. If the functor of one variable $\Theta(\ ,M)$ vanishes, then M is called a Θ-injective; Θ-projectives are defined dually. The E-functor Θ defines a relative homological algebra if there are sufficient Θ-injectives or Θ-projectives. We usually assume that there are sufficient Θ-injectives. Then every object in the category has a resolution by Θ-injective objects built up of simple extensions belonging to Θ. The 'Θ-injective resolutions' have the usual lifting property, that is, morphisms of the objects can be lifted to morphisms of their resolutions that are unique up to homotopy. Furthermore, if Φ is an E-functor contained in Θ with sufficient injectives, then the resolutions of § 8 beginning as Θ-injective and finishing as Φ-injective resolutions also have the lifting property. To simplify the discussion of their properties we define in § 3 more general resolutions of an abelian category. They are classes of resolutions which satisfy conditions ensuring that the lifting property holds for resolutions of objects and normal resolutions of complexes; in particular the class of injective (projective) resolutions is a resolution of the category if there are sufficient injectives (projectives). The only other examples that we can construct are obtained from nests of E-functors.

A resolution K of a category determines three sets of invariant functors of a given functor T, the K-derived functors, K-satellites, and K-cosatellites and also a sequence of natural transformations between them, the K-sequence. This sequence has order two and is exact at the K-satellites and K-cosatellites. It gives the relations between satellites and derived functors obtained by Cartan & Eilenberg (1956) for projective and injective resolutions, and the 'realization technique' of Zeeman (1957) can be used to show that it captures all the invariant information in the complexes obtained by applying T to K.

We show in § 5 that a functor T and two resolutions K and L, such that every member of K has a double complex resolution by L, determine an exact couple functor $(K, L)\ T$. In § 6 we consider a functor T of two variables—not necessarily in the same category—and compare the functor $(K, L)\ T$ obtained by regarding T as a family of functors in one variable with exact couple functors obtained by resolving both variables. The theorems obtained in § 6 which give conditions for the two types of exact couple to be isomorphic are called 'shifting theorems'. The results of §§ 5, 6 are applied in §§ 11, 12 to the resolutions associated with E-functors Θ and Φ such that Θ contains Φ. The relations between the exact couples are much simpler when the operations of forming satellites with respect to Θ and Φ commute; in particular the hypotheses of the shifting theorems are satisfied. In § 2 we define Φ to be central in Θ if its two products with Θ in the ringoid of Θ are equal, and in § 10 this is shown to be necessary and sufficient for the operations of forming satellites to commute. The exact couples of § 12 give for group theory spectral sequences relating the homology of a group with its derived functors relative to a subgroup: for a normal subgroup they are isomorphic to the spectral sequences of Hochschild & Serre (1953). For sheaf theory the exact couples determine spectral sequences reducing to a pair given in the Séminaire Chevalley (1958/59) for coherent sheaves over algebraic varieties.

Our only method of obtaining resolutions is described in §§ 13 to 15. We show that a pair of adjoint functors between two categories, one of which is abelian, determines an E-functor on the abelian category, and obtain conditions for this E-functor to have sufficient injectives. This construction is used to obtain two E-functors on a category of sheaves (§§ 17, 18) and the relative homological algebra of Hochschild (1956) for modules (§ 24). The results of § 15 are also used in § 21 to discuss several concepts of purity of submodules. Some simple conditions for an E-functor determined by a pair of adjoint functors to be central in a given E-functor are obtained in § 16, and used in §§ 24, 26 to discuss the 'Hochschild E-functors'.

We define the centre of an abelian category in § 19, and use it to construct E-functors in §§ 19, 22. When the category is 'hereditary' (that is, Ext^2 vanishes), some of these E-functors coincide with E-functors obtained from adjoint functors. The consequent existence theorems for projectives and injectives obtained in § 20 generalize some results of Nunke (1959) on Dedekind domains. § 23 contains an isolated result generalizing a theorem of Baer's (1958) and provides an example of an E-functor on the category of abelian groups without sufficient projectives or injectives. In § 27 we show that two elements of the centre acting as orthogonal idempotents on an E-functor determine direct sum decompositions of its derived functors, satellites, and cosatellites. This result is applied in § 28 to show that complement of the p-component of the homology of a finite group is isomorphic to the homology of the group relative to any Sylow p-subgroup.

1. E-functors

In this paper we shall use, for the most part, the terminology of Grothendieck (1957) for categories. Thus a category $\mathfrak{C}$ will be a class of objects $A, B, \ldots$, together with a set $\mathrm{Hom}_{\mathfrak{C}}(A, B)$—denoted by $\mathrm{Hom}(A, B)$ when there is no danger of confusion—of morphisms $\alpha, \beta, \ldots$, for each ordered pair of objects, an associative law of composition of morphisms, and an identity morphism 1_A in each set $\mathrm{Hom}(A, A)$. An abelian category will be a category in which finite direct sums and products are defined; each set Hom (A,B) is an abelian group, composition of morphisms is bilinear, and a zero object exists; cokernels and kernels are defined, and any morphism α has a canonical factorization $\alpha = (\text{im}\,\alpha)\,\bar{\alpha}(\text{coim}\,\alpha)$, where $\bar{\alpha}$ is an isomorphism, coim α and im α are the coimage and image of α. We shall write ker α and coker α for the kernel and cokernel of α. Also we shall write Ker α and Im α for the 'sources' of ker α and im α, and Coker α and Coim α for the 'targets' of coker α and coim α.

At first we shall make no assumption about the existence of injectives or projectives. Instead we shall suppose that $\text{Ext}^r_{\mathfrak{C}}$ (more briefly, Ext^r) is defined as by Buchsbaum (1959). So $\text{Ext}^r(A,B)$ is a class of equivalence classes of r-fold extensions of A by B. The class $\text{Ext}^r(A,B)$ has the algebraic structure of an abelian group, and will be regarded as an abelian group. It will be convenient to think of Hom together with Ext as forming a graded ringoid, the multiplication in this ringoid being defined as follows: if x is in $\text{Ext}^r(A,B)$ and y is in $\text{Ext}^s(B,C)$ $(r,s>0)$, then yx is defined to be the Yoneda product of y and x; if ξ is in Hom (A,B) and y is in $\text{Ext}^s(B,C)$ $(s>0)$, then $y\xi$ is defined to be $\text{Ext}^s(\xi,1_C)\,y$; if x is in $\text{Ext}^r(A,B)$ $(r>0)$ and η is in Hom (B,C), then ηx is defined to be $\text{Ext}^r(1_A,\eta)$; if ξ is in Hom (A,B) and η is in Hom (B,C), then $\eta\xi$ is defined by composition of morphisms. The operation of addition in the ringoid is the addition in Hom and Ext, and will be denoted by $+$. It follows from these definitions that multiplication is associative and distributive over addition. Also we have the operation of forming the direct sum of two elements. This is always defined for elements of the same degree and we shall denote it by $\oplus$.

The multiplication in this ringoid gives a convenient method of describing the connecting homomorphisms in the exact connected sequences of Ext associated with a simple extension (i.e. short exact sequence) $0 \to B \to X \to A \to 0$. Write x for the class of this simple extension in $\text{Ext}^1(A,B)$. Then the connecting homomorphism

$$\partial_{Y,x}\colon \text{Ext}^r(Y,A) \to \text{Ext}^{r+1}(Y,B)$$

is given by $\partial_{Y,x}(a) = xa$. This definition of $\partial_{Y,x}$ differs in sign from the usual definition of connecting homomorphisms, but it is more convenient here. Similarly the connecting homomorphism

$$\partial_{x,Y}\colon \text{Ext}^r(B,Y) \to \text{Ext}^{r+1}(A,Y)$$

is given by $\partial_{x,Y}(a) = ax$. With these definitions the square

$$\begin{array}{ccc} \text{Ext}^r(B',A) & \to & \text{Ext}^{r+1}(B',B) \\ \downarrow & & \downarrow \\ \text{Ext}^{r+1}(A',A) & \to & \text{Ext}^{r+2}(A',B), \end{array}$$

associated with the given simple extension and a second simple extension

$$0 \to A' \to X' \to B' \to 0$$

is commutative. For the images of an element x of $\text{Ext}^r(B',A)$ are $(a'x)\,a$ and $a'(xa)$, and the associative laws show that they coincide.

For each A, B in $\mathfrak{C}$ let $\Theta(A,B)$ be a non-empty subclass of $\text{Ext}^1(A,B)$. We say that Θ is a *natural class of simple extensions* (briefly, a *natural class*) if for any pair of morphisms $\alpha\colon A' \to A$ and $\beta\colon B \to B'$ in $\mathfrak{C}$, the restriction $\Theta(\alpha,\beta)$ of $\text{Ext}^1(\alpha,\beta)$ to $\Theta(A,B)$ has values in $\Theta(A',B')$. We regard a natural class as a functor—not necessarily additive—of two variables, which is contravariant in the first variable and covariant in the second variable. When the values of a natural class are subgroups, we call it an *E-functor*. Since Ext^1 is additive, an E-functor is additive.

Write $\tilde{\Theta}(A, B)$ for the class of simple extensions representing elements of $\Theta(A, B)$, and $\tilde{\Theta}$ for the union of the $\tilde{\Theta}(A, B)$. Let Θ be an E-functor. If $x \in \Theta(A, B)$ and

$$0 \to B \xrightarrow{\beta} X \xrightarrow{\alpha} A \to 0$$

is a simple extension representing x, then the connecting homomorphism $\partial_{Y,x}$ has image contained in $\Theta(Y, B)$, since $\partial_{Y,x}(\eta) = x\eta$. So we have the sequence

$$0 \to \mathrm{Hom}\,(Y, B) \to \mathrm{Hom}\,(Y, X) \to \mathrm{Hom}\,(Y, A) \to \Theta(Y, B) \to \Theta(Y, X) \to \Theta(Y, A).$$

By a similar argument we obtain

$$0 \to \mathrm{Hom}\,(A, Z) \to \mathrm{Hom}\,(X, Z) \to \mathrm{Hom}\,(B, Z) \to \Theta(A, Z) \to \Theta(X, Z) \to \Theta(B, Z).$$

The sequences are exact as far as $\Theta(Y, B)$ and $\Theta(A, Z)$, since Θ is a subgroup of Ext^1 and Ext is an exact connected sequence of functors. Also the product of any consecutive pair of morphisms in each sequence is zero, but it is not in general true that the sequences are exact at $\Theta(Y, X)$ and $\Theta(X, Z)$. If it is true, we shall call Θ a *closed E-functor*. That is to say Θ is closed if it is half exact in both variables on sequences belonging to $\tilde{\Theta}$. We shall also say that Θ is *closed on the right* (*left*) if it is exact in the second (first) variable on sequences belonging to $\tilde{\Theta}$.

Buchsbaum (1959) has shown that an 'h.f. class of monomorphisms' determines a closed E-functor. Conversely it may be shown that the class of monomorphisms determined by the simple extensions in $\tilde{\Theta}$ is an h.f. class when Θ is a closed E-functor. However, to show how the concepts of E-functor and h.f. class are related it will be more convenient to give a different—but equivalent—definition. Define an h.f. class to be a class M of morphisms such that:

(*a*) M contains all zero monomorphisms and epimorphisms;

(*b*) if $\alpha \in M$ and β is equivalent to α (i.e. $\alpha = \sigma\beta\tau$ where σ and τ are isomorphisms), then $\beta \in M$;

(*c*) $\alpha \in M$ if and only if $\ker \alpha$ and $\mathrm{coker}\, \alpha \in M$;

(*d*) if $\beta(\alpha)$ and $\alpha\beta$ are monomorphisms (epimorphisms) and $\alpha\beta \in M$, then $\beta \in M$ ($\alpha \in M$);

(e_1) if $\alpha, \beta \in M$ are monomorphisms and $\alpha\beta$ is defined, then $\alpha\beta \in M$;

(e_2) if $\alpha, \beta \in M$ are epimorphisms and $\alpha\beta$ is defined, then $\alpha\beta \in M$.

If M satisfies only (*a*), (*b*), (*c*), and (*d*), we call it an *f. class*.

Let M be a class of morphisms—not necessarily an f. class. Define $\Theta(A, B)$ to be the subset of $\mathrm{Ext}^1\,(A, B)$ whose members can be represented by simple extensions

$$0 \to B \xrightarrow{\beta} X \xrightarrow{\alpha} A \to 0,$$

where $\alpha, \beta \in M$. We notice that if M satisfies (*a*), (*c*), then both of α, β are in M when one of them is.

Proposition 1·1. *If M is an f. class, then Θ is a natural class.*

Proof. To prove that Θ is a natural class it is sufficient to show that if $x \in \Theta(A, B)$ and $\rho, \sigma \in \mathrm{Hom}\,(Y, A)$, $\mathrm{Hom}\,(B, Z)$ respectively, then $\sigma x \rho \in \Theta(Y, Z)$. First we show that $x\rho \in \Theta(Y, B)$. Let the simple extension

$$0 \to B \xrightarrow{\omega} W \to Y \to 0$$

represent $x\rho$. By the construction of $x\rho$ there is a morphism γ in $\mathrm{Hom}(W, X)$ such that $\gamma\omega = \beta$. Since β is a monomorphism in M and ω is a monomorphism, it follows from (d) that ω is in M. So (a) and (c) imply that $x\rho \in \Theta(Y, B)$. A dual argument shows that $\sigma x\rho \in \Theta(Y, Z)$.

Conversely, let $\Theta(A, B)$ be a subclass of $\mathrm{Ext}^1(A, B)$ defined for all A, B. Write $\tilde{\Theta}(A, B)$ for the class of simple extensions representing elements of $\Theta(A, B)$, and $\tilde{\Theta} = \bigcup \tilde{\Theta}(A, B)$. Define M to be the class of morphisms containing all the epimorphisms and monomorphisms of all the simple extensions in $\tilde{\Theta}$, and all morphisms whose kernels and cokernels belong to the simple extensions in $\tilde{\Theta}$.

Proposition 1·2. *If Θ is a natural class, then M is an f. class.*

Proof. Since $\Theta(A, 0)$, $\Theta(0, A)$ are non-empty, (a) is satisfied. By the construction of M, (b) and (c) are satisfied. It remains to be seen that (d) is satisfied. Let $\beta: B \to A$ be a monomorphism and $\alpha: A \to C$ be a morphism such that $\gamma = \alpha\beta$ is a monomorphism in M. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & B & \xrightarrow{\beta} & A & \to & D & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta} & & \\ 0 & \to & B & \xrightarrow{\gamma} & C & \to & E & \to & 0, \end{array}$$

where δ is induced by α, and D, E are $\mathrm{Coker}\,\beta$, $\mathrm{Coker}\,\gamma$ respectively. Since $\gamma \in M$, the lower simple extension is in $\tilde{\Theta}(E, B)$. But the upper simple extension is induced from it by δ, and Θ is a natural class; so it is in $\tilde{\Theta}(D, B)$. Hence $\beta \in M$,and (d) is satisfied for monomorphisms. By duality (d) is satisfied for epimorphisms.

Now we show that the constructions of Θ from M, and M from Θ are mutually inverse for f. classes and natural classes. First let Θ be a given natural class. Write M for the f. class constructed from Θ, and Θ' for the natural class constructed from M. Then a simple extension is in $\tilde{\Theta}'$ if and only if its monomorphism is in M. But the monomorphisms of M are just the monomorphisms of the simple extensions in $\tilde{\Theta}$. So $\tilde{\Theta} = \tilde{\Theta}'$, and $\Theta = \Theta'$. Secondly let M be a given f. class. Write Θ for the natural class constructed from M, and M' for the f. class constructed from Θ. Axioms (a) and (c) show that $M = M'$ if they contain the same monomorphisms. By construction the monomorphisms in M' are the same as the monomorphisms in the simple extensions in $\tilde{\Theta}$, and the construction of Θ from M together with axiom (b) shows that the monomorphisms in M are also the same as the monomorphisms in the simple extensions in $\tilde{\Theta}$. Hence $M = M'$. So we have shown that there is a one-one correspondence between f. classes and natural classes. We shall call a morphism that belongs to the f. class defined by a natural class Θ a Θ-morphism.

Next we show that an E-functor Θ is closed if and only if the Θ-morphisms form an h.f. class. More precisely we have:

Theorem 1·1. (i) *The natural class Θ is an E-functor if the Θ-morphisms satisfy (e_1) or (e_2).*

(ii) *The E-functor Θ is closed on the right if and only if the class of Θ-morphisms satisfies (e_1), and it is closed on the left if and only if the class of Θ-morphisms satisfies (e_2).*

Proof. First we prove (i). Suppose that the Θ-morphisms satisfy (e_1), and let x, y be in $\Theta(A, B)$. Since $x+y$ is induced from $x \oplus y$ by the codiagonal morphism $B \oplus B \to B$ and the diagonal morphism $A \to A \oplus A$, and Θ is a natural class, it is sufficient to show that $x \oplus y$ is in Θ. Let x and y have representatives

$$0 \to B \xrightarrow{\xi} X \to A \to 0, \quad 0 \to B \xrightarrow{\eta} Y \to A \to 0.$$

The monomorphism ζ in $x \oplus y$ is $(\xi \oplus 1_Y)(1_B \oplus \eta)$. Left multiplication of $1_B \oplus \eta$ by $0 \oplus 1_Y$ shows that $1_B \oplus \eta$ is a right factor of η. So $1_B \oplus \eta$ is a Θ-monomorphism by axiom (d). Similarly $\xi \oplus 1_Y$ is a Θ-monomorphism. Hence (e_1) shows that ζ is a Θ-monomorphism. By duality (e_2) also implies that Θ is an E-functor. So (i) is proved.

Now we prove (ii). It is sufficient to prove the first statement, the second follows by duality.

Suppose that the class of Θ-morphisms satisfies (e_1). Let

$$0 \to B \xrightarrow{\beta} X \xrightarrow{\alpha} A \to 0$$

belong to $\tilde{\Theta}(A, B)$, and write x for its image in $\Theta(A, B)$. To prove that Θ is closed on the right we have to show that, if $y \in \Theta(Y, X)$ and $\alpha y = 0$, then there exists z in $\Theta(Y, B)$ such that $y = \beta z$. Since Ext^1 is half-exact, there exists z in $\text{Ext}^1(Y, B)$ such that $y = \beta z$. So we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & B & \xrightarrow{\gamma} & Z & \to & Y & \to 0 \\ & \downarrow \beta & & \downarrow \zeta & & & \\ 0 \to & X & \xrightarrow{\xi} & W & \to & Y & \to 0, \end{array}$$

where the first row represents z and the second represents y. Now β is a Θ-monomorphism and ξ is a Θ-monomorphism, since x and y are in $\Theta(A, B)$ and $\Theta(Y, X)$. So by (e_1), $\xi\beta$ is a Θ-monomorphism. But $\zeta\gamma = \xi\beta$, and γ is a monomorphism. So (d) shows that γ is a Θ-monomorphism. Hence $z \in \Theta(Y, B)$. So Θ is closed on the right.

Now suppose that Θ is closed on the right. Let α in $\text{Hom}(B, A)$ and β in $\text{Hom}(C, B)$ be Θ-monomorphisms. Put $\gamma = \alpha\beta$. Then we have a commutative diagram

$$\begin{array}{ccccc} C & = & C & & \\ \downarrow \beta & & \downarrow \gamma & & \\ B & \xrightarrow{\alpha} & A & \xrightarrow{\alpha'} & F, \\ \downarrow \beta' & & \downarrow \gamma' & & \| \\ D & \xrightarrow{\delta} & E & \xrightarrow{\delta'} & F, \end{array}$$

where $\alpha' = \text{coker}\,\alpha$, etc., and δ is the monomorphism induced by α. To prove that γ is in M we need the following lemma which we call the *eight-lemma*, and whose proof we merely outline.

EIGHT-LEMMA. *Let* $X \in \mathfrak{C}$. *By applying* Hom $(X, \ )$ *to the above diagram we obtain two homomorphisms of* Hom (X, E) *into* $\text{Ext}^1(X, B)$, *namely*

$$\text{Hom}(X, E) \to \text{Hom}(X, F) = \text{Hom}(X, F) \to \text{Ext}^1(X, B),$$

and

$$\text{Hom}(X, E) \to \text{Ext}^1(X, C) = \text{Ext}^1(X, C) \to \text{Ext}^1(X, B).$$

Then these homomorphisms are identical.

If $\mathfrak{C}$ has enough injectives, this may be proved by taking an injective resolution of the diagram, and 'element chasing' in the corresponding diagram of complexes obtained after applying Hom $(X, \ )$. If $\mathfrak{C}$ has enough projectives, we take a projective resolution P of X and apply Hom $(P, \ )$ to the diagram. Again the result follows by element chasing. In the general case when there are not sufficient injectives or projectives the lemma may be proved by a direct construction.

To apply the lemma put $X = E$, and let a, c be the elements of $\text{Ext}^1(F, B)$ and $\text{Ext}^1(E, C)$ representing the second row and column of the diagram. The connecting homomorphism $\text{Hom}(E, F) \to \text{Ext}^1(E, B)$ is given by $\xi \to a\xi$. So the image of 1_E under the first homomorphism is $a\delta'$. Similarly the image of 1_E under the second homomorphism is βc. Hence $a\delta' = \beta c$. Since $a \in \Theta(F, B)$, and $\delta' \in \text{Hom}(E, F)$, it follows that $\beta c \in \Theta(E, B)$. Now the sequence

$$\Theta(E, C) \xrightarrow{\beta} \Theta(E, B) \xrightarrow{\beta'} \Theta(E, D)$$

is exact, for Θ is closed on the right and $0 \to C \to B \to D \to 0$ is in $\tilde{\Theta}(C, D)$. Furthermore $\beta'\beta c = 0$. Hence there exists c' in $\Theta(E, C)$ such that $\beta c = \beta c'$, that is $\beta(c - c') = 0$. Since the sequence

$$\text{Hom}(E, D) \to \text{Ext}^1(E, C) \xrightarrow{\beta} \text{Ext}^1(E, B)$$

is exact, $c - c' = b\eta$ where $\eta \in \text{Hom}(E, D)$ and b is the representative of $0 \to C \to B \to D \to 0$ in $\Theta(D, C)$. So $c - c' \in \Theta(E, C)$. But $c' \in \Theta(E, C)$. Hence c is in $\Theta(E, C)$, and consequently γ is a Θ-morphism. So the class of Θ-morphisms satisfies (e_1) and this proves the theorem.

As an application of this theorem we prove a simple result about families of E-functors. Let Θ_i $(i \in I)$ be a family of E-functors. Define their intersection Θ by $\Theta(A, B) = \bigcap_{i \in I} \Theta_i(A, B)$. Then Θ is an E-functor.

Proposition 1·3. *If Θ_i are closed E-functors, then Θ is a closed E-functor.*

Proof. Let M_i be the h.f. class corresponding to Θ_i. The axioms for h.f. classes show at once that $\bigcap M_i$ is an h.f. class. So Θ is closed.

2. The ringoid of an E-functor

In this section we collect together some simple properties of an E-functor Θ which have been obtained by Buchsbaum (1959) in the general case or by Yoneda (1956) for Ext^1. We also introduce the idea of a central E-functor.

First we recall the definition of Θ^n. An n-fold Θ-extension of A by B is an exact sequence of Θ-morphisms

$$0 \to B \to X^0 \to X^1 \to \ldots \to X^n \to A \to 0. \qquad (\tilde{x})$$

Two such extensions $\tilde{x}$ and $\tilde{y}$ are said to be similar if there is a complex morphism $\tilde{x} \to \tilde{y}$ or $\tilde{y} \to \tilde{x}$ extending the identity morphisms on A and B. Furthermore, $\tilde{x}$ and $\tilde{y}$ are said to be equivalent if there is a sequence of n-fold Θ-extensions $\tilde{z}_0$ $(= \tilde{x})$, $\tilde{z}_1, \ldots, \tilde{z}_r$ $(= \tilde{y})$ such that $\tilde{z}_i$ is similar to $\tilde{z}_{i+1}$. Then $\Theta^n(A, B)$ is defined to be the class of equivalence classes of n-fold Θ-extensions of A by B. In particular $\Theta^1 = \Theta$. We do not know that Θ^n is a set, but here this is not important. When Θ^n is used in § 6 *et seq.*, it will be obtained under hypotheses which ensure that it is a set.

Buchsbaum (1959) shows that Θ^n is a functor covariant in the second variable and contravariant in the first variable, and $\Theta^n(A, B)$ has a natural abelian group structure (the sum $x + y$ of x, y in $\Theta^n(A, B)$ is the image of their direct sum under the diagonal morphisms

$A \to A \oplus A$ and $B \oplus B \to B$). If $x \in \Theta^p(A, B)$ and $y \in \Theta^q(B, C)$, their product $y \cdot x$ is the element of $\Theta^{p+q}(A, C)$ represented by the $(p+q)$-fold extension obtained by 'splicing' representatives of y and x at B. This product is bilinear and associative. If x is in $\Theta^1(A, B)$ and y is in $\Theta^1(B, C)$, then two products are defined, namely yx in $\mathrm{Ext}^2(C, A)$ and $y \cdot x$ in $\Theta^2(C, A)$, and these two products must not be confused. In fact yx is the image of $y \cdot x$ under the natural transformation $\Theta^2 \to \mathrm{Ext}^2$ which is not in general an injection.

We shall frequently need the following result which we state without proof:

LEMMA 2·1. *If Θ is a closed E-functor, the diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & W & \to & B & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Y & \to & M & \to & X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & D & \to & Z & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

is commutative and exact, and all its morphisms are Θ-morphisms, then $w \cdot x = -y \cdot z$, where w is the class of $0 \to A \to W \to B \to 0$ in $\Theta(B, A)$, etc.

Write Θ^0 for Hom, and let $0 \to B \to X \to A \to 0$ be a representative of an element x of $\Theta(A, B)$. Then we have a sequence of order two,

$$\to \Theta^n(Y, B) \to \Theta^n(Y, X) \to \Theta^n(Y, A) \to \Theta^{n+1}(Y, B) \to,$$

where the connecting homomorphism mapping Θ^n into Θ^{n+1} is defined by $a \to x \cdot a$ for $n > 0$, and $\alpha \to x\alpha$ for $n = 0$. We obtain a similar sequence by interchanging the variables. When Θ is a closed E-functor both of these sequences are exact.

Let Φ be an E-functor such that $\Phi \subset \Theta$, that is $\Phi(A, B) \subset \Theta(A, B)$ for all A, B in $\mathfrak{G}$. Define $\Theta \cdot \Phi(A, B)$ to be the subclass of $\Theta^2(A, B)$ formed by the products $y \cdot x$ with x in $\Phi(A, C)$ and y in $\Theta(C, B)$, for some C in $\mathfrak{C}$. Then $\Theta \cdot \Phi$ is a functor, covariant in the second variable and contravariant in the first variable, and the inclusion $\Theta \cdot \Phi \to \Theta^2$ is a natural transformation of functors. This implies that $\Theta \cdot \Phi(A, B)$ is a subgroup of $\Theta^2(A, B)$. For let $u, v \in \Theta \cdot \Phi(A, B)$. Then $-u = (-1_B) u$ is in $\Theta \cdot \Phi(A, B)$. Also $u \oplus v$ is in $\Theta \cdot \Phi(A \oplus A, B \oplus B)$, and $u + v$ is induced from $u \oplus v$ by the diagonal and codiagonal morphisms $A \to A \oplus A$ and $B \oplus B \to B$. Thus $u + v$ is in $\Theta \cdot \Phi(A, B)$. So $\Theta \cdot \Phi(A, B)$ is a subgroup of $\Theta^2(A, B)$.

Similarly define $\Phi \cdot \Theta(A, B)$ to be the subclass of $\Theta^2(A, B)$ formed by the products $x \cdot y$ with x in $\Phi(C, B)$ and y in $\Theta(A, C)$ for some C in $\mathfrak{C}$. Then $\Phi \cdot \Theta$ is a functor covariant in the second variable and contravariant in the first variable. As before $\Phi \cdot \Theta(A, B)$ can be shown to be a subgroup of $\Theta^2(A, B)$.

We shall say that Φ *is central in* Θ if $\Phi \cdot \Theta = \Theta \cdot \Phi$. If Φ is central in Ext^1, then we shall call Φ a *central* E-functor. We shall see later that pairs of *E*-functors one of which is central in the other have very simple properties and occur naturally in the theories of the homology of a group relative to a subgroup and of the homology of coherent sheaves. In § 15 it is shown that there exist E-functors which are not central in Ext^1.

3. Resolutions of categories

We follow Cartan & Eilenberg (1956) and use complex to mean cochain complex unless the contrary is stated. Let X^* denote a complex. We write δ_X (or just δ) for its differentiation, and δ^n for the component of δ in Hom (X^n, X^{n+1}). We shall say that X^* is a *right resolution* of A if $X^n = 0$ for $n < 0$, Im $\delta^n = \operatorname{Ker} \delta^{n+1}$ for $n \geqslant 0$, and $\operatorname{Ker} \delta^0 \cong A$.

We recall that an object M is said to be injective over a monomorphism α in Hom (A, B) if each morphism in Hom (A, M) has α as a right factor.

A class K of right resolutions of objects in a category $\mathfrak{C}$ is called a *right resolution* of $\mathfrak{C}$ if: (i) each object of $\mathfrak{C}$ has a resolution in K (or, more briefly, a K-resolution); (ii) the zero resolution $\ldots \to 0 \to 0 \to \ldots$ belongs to K; (iii) for each pair X^*, Y^* of K-resolutions, Y^n is injective over ker δ^n_X and im δ^n_X for all n. Left resolutions of $\mathfrak{C}$ are defined dually. If $\mathfrak{C}$ has enough injectives, the class of injective resolutions forms a right resolution of $\mathfrak{C}$.

Let K be a right resolution of $\mathfrak{C}$. The proof of Prop. 1·1, p. 76, Cartan & Eilenberg (1956) with only verbal modifications shows that:

Proposition 3·1. *If X^*, Y^* are K-resolutions of A, B and $\alpha \in$ Hom (A, B), there exists a complex morphism $\alpha^* : X^* \to Y^*$ covering α. If β^* is also a complex morphism of X^* into Y^* covering α, there exists a homotopy σ: $\alpha^* \simeq \beta^*$ (that is, a sequence σ of morphisms σ^n in* Hom (X^n, Y^{n-1}) *such that $\alpha^* - \beta^* = \delta_Y \sigma + \sigma \delta_X$).*

This property of K-resolutions makes it possible to define derived functors in the usual way, and satellites and cosatellites in the way suggested by Cartan & Eilenberg (1956, ex. 1, p. 104). The introduction of cosatellites enables us to relate these three types of functor by means of a sequence of natural transformations. First we need some more notation.

Write μ_X and ϵ_X for the kernel and cokernel of the differentiation δ_X of a complex X^*. Since $\delta^2 = 0$, there is a monomorphism ι_X such that $\delta^n_X = \mu^{n+1}_X \iota^{n+1}_X \epsilon^n_X$. Write $\overline{X}^n$ for Ker δ^n_X ($=$ Im ι^n_X), and η^{n+1}_X for $\iota^{n+1}_X \epsilon^n_X$. If X^* is a right resolution of A, then ι^n_X is an isomorphism for $n \neq 0$, the zero monomorphism into A for $n = 0$, and $\overline{X}^0 = A$. It will be convenient to regard the sequence $\{\overline{X}^n\}$ as a complex $\overline{X}^*$ with zero differentiation.

Let X^*, Y^* be K-resolutions of A, B, α be a morphism of A into B, and α^* be a complex morphism of X^* into Y^* covering α. Since η^n_X is an epimorphism for $n \geqslant 1$, α^* induces a unique morphism $\bar{\alpha}^*$: $\overline{X}^* \to \overline{Y}^*$ such that $\alpha^* \mu_X = \mu_Y \bar{\alpha}^*$ and $\eta_Y \alpha^* = \bar{\alpha}^* \eta_X$. Suppose that α is the zero morphism. By proposition 3·1 there exists a homotopy σ: $\alpha^* \simeq 0$. Since $\delta_X \mu_X = 0$

$$\mu_Y \bar{\alpha}^* = \alpha^* \mu_X = \delta_Y \sigma \mu_X.$$

But μ_Y is a monomorphism and $\delta_Y = \mu_Y \eta_Y$, so

$$\bar{\alpha}^* = \eta_Y \sigma \mu_X. \tag{3·1}$$

Now let T be a covariant functor on $\mathfrak{C}$ with values in an abelian category $\mathfrak{D}$. To obtain the following results for a contravariant functor one dualizes $\mathfrak{D}$.

Lemma 3·1. *Let X^*, Y^* be K-resolutions of A, B and α^* be a complex morphism of X^* into Y^* covering α in* Hom (A, B). *Then the morphisms*

$$H(TX^*) \to H(TY^*), \quad \operatorname{Ker} T(\mu_X) \to \operatorname{Ker} T(\mu_Y) \quad \textit{and} \quad \operatorname{Coker} T(\eta_X) \to \operatorname{Coker} T(\eta_Y)$$

induced by α^* are determined uniquely by α. (H denotes the operation of forming the homology complex.)

Proof. It is sufficient to show that α^* induces zero morphisms when α is the zero morphism. Suppose then that α is the zero morphism. There is a homotopy $\sigma: \alpha^* \simeq 0$ by proposition 3·1. So by a standard result of homology theory α^* induces the zero morphism from

$$H(TX^*) \to H(TY^*).$$

The morphism of Ker $T(\mu_X)$ into Ker $T(\mu_Y)$ induced by α^* is also induced by $\bar{\alpha}^*$. Since $\bar{\alpha}^* = \eta_Y \sigma \mu_X$, and T is covariant, the required morphism has $T(\mu_X)$ as a right factor. So it vanishes on Ker $T(\mu_X)$. Similarly the morphism induced on Coker $T(\eta_X)$ vanishes. Hence the lemma is proved.

Let X^* and Y^* be two K-resolutions of A, α^* be a complex morphism of X^* into Y^* covering 1_A, and β^* a complex morphism of Y^* into X^* covering 1_A. Then $\beta^*\alpha^*$ is a complex morphism of X^* into itself covering 1_A. Since the identity morphism on X^* also covers 1_A, lemma 3·1 shows that $\beta^*\alpha^*$ induces the identity morphisms on $H(TX^*)$, Ker $T(\mu_X)$, and Coker $T(\eta_X)$. Similarly $\alpha^*\beta^*$ induces the identity morphisms on $H(TY^*)$, Ker $T(\mu_Y)$, and Coker $T(\eta_Y)$. So $H(TX^*)$, Ker $T(\mu_X)$, and Coker $T(\eta_X)$ are determined up to isomorphism by K, T, and A. We denote them by KTA, $\hat{K}TA$, and $\check{K}TA$. Their components of degree n are given by

$$K^n TA = H^n(TX^*), \quad \hat{K}^n TA = \mathrm{Ker}\, T(\mu_X^n), \quad \check{K}^n TA = \mathrm{Coker}\, T(\eta_X^n).$$

It follows that they vanish for $n < 0$, $\check{K}^0 T = T$, and $\hat{K}^0 T$ is the kernel of the morphism of TA into TX^0 induced by the augmentation of X^*.

Now let X^*, Y^* be K-resolutions of A, B. If α is in Hom (A, B), then there exists a complex morphism α^* of X^* into Y^* covering α. By lemma 3·1 the induced morphisms of KTA, $\hat{K}TA$, and $\check{K}TA$ into KTB, $\hat{K}TB$, and $\check{K}TB$ respectively are determined by α. Denote them by $KT(\alpha)$, $\hat{K}T(\alpha)$, and $\check{K}T(\alpha)$. If $B = A$ and α is the identity on A, we may choose α^* to be the identity of X^*. So $KT(1_A) = 1_{KTA}$, etc. If $\beta \in$ Hom (B, C), it is trivial to verify that $KT(\beta)\, KT(\alpha) = KT(\beta\alpha)$, etc. Hence for each n, K^nT, $\hat{K}^nT$, and $\check{K}^nT$ are covariant functors from $\mathfrak{C}$ to $\mathfrak{D}$. We call them the *K-derived functors*, *K-cosatellites*, and *K-satellites* of T.

If K is the class of injective resolutions of a category $\mathfrak{C}$ with sufficient injectives, the K-satellites of T are the right satellites of Cartan & Eilenberg (1956).

Let X^* be a K-resolution of an object A in $\mathfrak{C}$. Then it may be verified that there is a commutative and exact diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & KTA & & \hat{K}TA & & \\
 & & {\scriptstyle [\eta_X]}\downarrow & & \downarrow & & \\
0 & \to & \mathrm{Im}\, T(\eta_X) & \to & TX^* & \to & \check{K}TA \to 0 \\
 & & {\scriptstyle [\mu_X]}\downarrow & & {\scriptstyle [\mu_X]}\downarrow & & \\
0 & \to & \mathrm{Im}\, T(\delta_X) & \to & \mathrm{Ker}\, T(\delta_X) & \to & KTA \to 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array} \qquad (3·2)$$

in which the morphisms are either induced by the morphisms in brackets or are canonical inclusions or projections. The diagram determines a sequence of morphisms

$$KTA \to \hat{K}TA \to \check{K}TA \to KTA,$$

which we shall denote by $\eta(A)$, $\iota(A)$, and $\mu(A)$ respectively. The exactness of the diagram shows that the sequence is exact, and a calculation of degrees shows that $\eta(A)$ has degree 1, and $\iota(A)$, $\mu(A)$ have degree 0. If Y^* is a K-resolution of an object B in $\mathfrak{C}$ and α^* is a complex morphism of X^* into Y^* covering α in Hom (A, B), then we obtain a similar diagram from Y^*. The two diagrams joined by the morphisms induced by α^* form a commutative diagram. When $B = A$ and $\alpha = 1_A$, this shows that $\eta(A)$, etc., are independent of the choice of K-resolution X^*, and then the diagram for general A, B, and α shows that η, ι, and μ are natural transformations of functors. So the kernels and images of η, ι and μ are functors. Since η_X and μ_X induce $\eta(A)$ and $\mu(A)$, we have $\eta(A)\,\mu(A) = 0$. Define the *K-excess* of T to be $\operatorname{Ker}\eta/\operatorname{Im}\mu$. The K-excess of T is a functor and we denote it by EKT. From the diagram (3·2), $\operatorname{Ker}\eta(A) = \operatorname{Ker} T(\eta_X)/\operatorname{Im} T(\delta_X)$ and $\operatorname{Im}\mu(A) = \operatorname{Im} T(\mu_X)/\operatorname{Im} T(\delta_X)$. Therefore we have the formula

$$EKTA = \operatorname{Ker} T(\eta_X)/\operatorname{Im} T(\mu_X). \tag{3·3}$$

So we have shown that associated with each right resolution K of $\mathfrak{C}$ and each covariant functor T from $\mathfrak{C}$ to $\mathfrak{D}$ there is a sequence of satellites $\check{K}^n T$, cosatellites $\hat{K}^n T$, and derived functors $K^n T$, connected by natural transformations of functors η^n, ι^n, μ^n such that

$$0 \to \hat{K}^0 T \xrightarrow{\iota^0} T \xrightarrow{\mu^0} K^0 T \xrightarrow{\eta^1} \hat{K}^1 T \to \dots \to K^{n-1} T \xrightarrow{\eta^n} \hat{K}^n T \xrightarrow{\iota^n} \check{K}^n T \xrightarrow{\mu^n} K^n T \to \dots$$

has order two, and is exact at the satellites and cosatellites. This sequence will be called the *K-sequence* of T. Also we can prove that a natural transformation of functors $\omega\colon T \to U$ induces natural transformations of the K-sequences of T and U.

Consider now two right resolutions K and L of $\mathfrak{C}$. We say that K *dominates* L, and write $K \succ L$, if for any X^* in K and Y^* in L the object X^n is injective over $\ker\delta_Y^n$ and $\operatorname{im}\delta_Y^n$. In particular the class of injective resolutions—if it exists—dominates all other classes of right resolutions. By the arguments of Cartan & Eilenberg (1956, p. 76) we may prove:

PROPOSITION 3·2. *If K dominates L, X^* is a K-resolution of A, and Y^* is an L-resolution of B, then any morphism α in* Hom (B, A) *can be covered by a complex morphism α^* of Y^* into X^*, and any two complex morphisms covering α are homotopic.*

Therefore when $K \succ L$ there are natural transformations of functors

$$\tau_{K,L}\colon LT \to KT, \quad \hat{\tau}_{K,L}\colon \hat{L}T \to \hat{K}T, \quad \check{\tau}_{K,L}\colon \check{L}T \to \check{K}T,$$

and they determine a natural transformation of the L-sequence of T into the K-sequence of T (that is, they commute with η, ι, μ). Furthermore, if $\omega\colon T \to U$ is a natural transformation of functors, then $\tau_{K,L}$, etc., commute with the extensions of ω to derived functors, etc. We shall call $\tau_{K,L}$, $\hat{\tau}_{K,L}$, and $\check{\tau}_{K,L}$, *τ-transformations* of functors, and their values in the class of morphisms of $\mathfrak{D}$, *τ-morphisms*.

4. K-resolutions of complexes

In the previous section we have considered K-resolutions of objects: in this section we define K-resolutions of complexes, prove the fundamental existence and invariance theorems and obtain connected sequences of K-derived functors, K-satellites and K-cosatellites for special simple extensions. Again most of the proofs are simple modifications of analogous results of Cartan & Eilenberg (1956) and we shall not give the details.

Let M^{**} be a double complex with two commuting differentiations δ_{M1}, δ_{M2} (or just δ_1, δ_2) of types $(1,0)$, $(0,1)$. Write $Z_i(M)$, $B_i(M)$, $Z'_i(M)$, $B'_i(M)$, and $H_i(M)$ for $\operatorname{Ker}\delta_i$, $\operatorname{Im}\delta_i$, $\operatorname{Coker}\delta_i$, $\operatorname{Coim}\delta_i$, and $\operatorname{Ker}\delta_i/\operatorname{Im}\delta_i$; they are to be regarded as complexes with differentiation induced by δ_j $(j \neq i)$. If $M^{rs} = 0$ for $s < 0$, then M together with a monomorphism of complexes $\mu^*: X^* \to M^{*0}$ is called a *right complex over X^* with augmentation μ^**. We say that M^{**} is a right resolution of X^* if the complexes M^{p*}, $Z_1^{p*}(M)$, etc., are right resolutions of X^p, $Z^p(X)$, etc. If also $\ker\delta_1$ and $\operatorname{coker}\delta_1$ split, we call M^{**} a *normal resolution* of X^*.

Before defining K-resolutions of complexes in general we consider the case of a simple extension X^*; that is, a complex for which δ_X is exact and $X^i = 0$ for $i \neq 0, 1, 2$. We call a double complex M^{**} a K-resolution of X^* if it is a right complex over X^*, $M^{ij} = 0$ whenever $X^i = 0$, M^{i*} is a K-resolution of X^i for all i, and δ_1 is exact and splits. So a K-resolution of X^* is a normal resolution of X^*.

Proposition 4·1. *Let K and L be resolutions of $\mathfrak{C}$ such that $K \succ L$. If X^* and Y^* are simple extensions admitting, respectively, a K-resolution M^{**} and an L-resolution N^{**}, then any complex morphism $\alpha^*: Y^* \to X^*$ can be covered by a double complex morphism $\alpha^{**}: N^{**} \to M^{**}$ in which α^{0*} and α^{2*} may be any complex morphisms covering α^0 and α^2. If β^{**} is another double complex morphism covering α^* and σ^i $(i = 0, 2)$ are homotopies between α^{i*} and β^{i*} $(i = 0, 2)$, then there exists a homotopy σ^1 between α^{1*} and β^{1*} such that*

$$\begin{array}{ccccc} N^{0p} & \to & N^{1p} & \to & N^{2p} \\ \sigma^{0p}\downarrow & & \sigma^{1p}\downarrow & & \sigma^{2p}\downarrow \\ M^{0p-1} & \to & M^{1p-1} & \to & M^{2p-1} \end{array}$$

commutes.

Proof. The proof is the same as the proof of Prop. V, 2·3 of Cartan & Eilenberg (1956) with only verbal changes, for that proof depends only on the normality of the complexes, the solubility of equations (4) for γ_n, and the equations on p. 82 for t_n. The conditions of injectivity in the definition of the relation $K \succ L$ ensure that the equations are soluble.

As a first application of this result we obtain a connected sequence of K-derived functors for any simple extension admitting a K-resolution. Let M^{**} be a K-resolution of a simple extension X^*. Since M^{**} is normal, the columns of TM^{**} are exact, and we have an exact cohomology sequence for a covariant functor T,

$$0 \to K^0TX^0 \to K^0TX^1 \to K^0TX^2 \to K^1TX^0 \to, \quad \text{etc.}$$

By proposition 4·1 the connecting morphisms are independent of the choice of K-resolution M^{**}. Furthermore, if K dominates L, the simple extension Y^* admits an L-resolution, and α^* is a complex morphism of Y^* into X^*, then proposition 4·1 shows that the square

$$\begin{array}{ccc} LTY^2 & \to & LTY^0 \\ \tau_{K,L}(\alpha^2)\downarrow & & \downarrow\tau_{K,L}(\alpha^0) \\ KTX^2 & \to & KTX^0 \end{array}$$

is commutative.

We next describe briefly how connecting morphisms for the satellites and cosatellites can be constructed. The proofs are omitted since these morphisms are not referred to again.

Let M^{**} be a K-resolution of a simple extension X^*. Write δ^i for the differentiation δ_1^{i*} of M^{**}. Factorizing the rows of M^{**} as in § 3 gives a commutative diagram

$$\begin{array}{ccccc} \overline{M}^{0*} & \xrightarrow{\mu^0} & M^{0*} & \xrightarrow{\eta^0} & \overline{M}^{0*} \\ \pi^0\downarrow & & \delta^0\downarrow & & \pi^0\downarrow \\ \overline{M}^{1*} & \xrightarrow{\mu^1} & M^{1*} & \xrightarrow{\eta^1} & \overline{M}^{1*} \\ \pi^1\downarrow & & \delta^1\downarrow & & \pi^1\downarrow \\ \overline{M}^{2*} & \xrightarrow{\mu^2} & M^{2*} & \xrightarrow{\eta^2} & \overline{M}^{2*} \end{array}$$

in which: δ^i induces π^i; π^0, δ^0 are monomorphisms; π^1, δ^1 are epimorphisms; the columns are exact; and δ^0, δ^1 split. Let ϕ^0 and ϕ^1 be left and right inverses of δ^0 and δ^1, respectively, such that $\delta^0\phi^0+\phi^1\delta^1$ is the identity of M^{1*}. Put $\alpha = \phi^0\mu^1$ and $\beta = \eta^1\phi^1$. It is easy to verify that there is a morphism γ of $\overline{M}^{2*}$ into $\overline{M}^{0*}$ such that $\eta^0\alpha = -\gamma\pi^1$ and $\beta\mu^2 = \pi^0\gamma$. Then the connecting morphisms

$$\hat{\partial}\colon \hat{K}TX^2 \to \hat{K}TX^0 \quad \text{and} \quad \check{\partial}\colon \check{K}TX^2 \to \check{K}TX^0$$

are induced by $-\gamma$ and γ. These connecting morphisms have the usual properties of uniqueness and naturality; their associated connected sequence has order two, but it need not be exact. We can also show that $\alpha\beta$ induces the connecting morphism for derived functors. Finally, the connecting morphisms relate the K-sequences of TX^2 and TX^0 in the diagram

$$\begin{array}{ccccccc} KTX^2 & \to & \hat{K}TX^2 & \to & \check{K}TX^2 & \to & KTX^2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ KTX^0 & \to & \hat{K}TX^0 & \to & \check{K}TX^0 & \to & KTX^0, \end{array}$$

in which the end squares are commutative and the centre square is anticommutative.

We now return to the question of defining K-resolutions of complexes. The double complex M^{**} is called a *K-resolution* of X^* if: it is a right complex over X^*; M^{p*}, $Z_1^{p*}(M)$, $Z_1'^{p*}(M)$, $B_1^{p*}(M)$, $B_1'^{p*}(M)$, and $H_1^{p*}(M)$ are K-resolutions of X^p, $Z^p(X)$, $Z'^p(X)$, $B^p(X)$, $B'^p(X)$, and $H^p(X)$; M^{p*}, etc., are zero whenever X^p, etc., are zero; $\ker\delta_1$ and $\operatorname{coker}\delta_1$ split. So M^{**} is a K-resolution of X^* if and only if the complexes

$$0 \to B_1^{p*}(M) \to Z_1^{p*}(M) \to H_1^{p*}(M) \to 0, \quad 0 \to Z_1^{p*}(M) \to M^{p*} \to B_1'^{p*}(M) \to 0,$$

and their duals are K-resolutions of

$$0 \to B^p(X) \to Z^p(X) \to H^p(X) \to 0, \quad 0 \to Z^p(X) \to X^p \to B'^p(X) \to 0,$$

and their duals. The fundamental property of K-resolutions is:

Proposition 4·2. *If K and L are resolutions of $\mathfrak{C}$ such that $K \succ L$, and X^* and Y^* are complexes admitting respectively a K-resolution M^{**} and an L-resolution N^{**}, then any complex morphism $\alpha^*: Y^* \to X^*$ can be covered by a double complex morphism $\alpha^{**}: N^{**} \to M^{**}$, and any two double complex morphisms covering α^* are homotopic.*

Proof. The proof of this proposition is similar to the proof of the last part of Prop. 1·2, p. 365, Cartan & Eilenberg (1956). Since that proof depends only on Prop. 1·2, p. 77 and Prop. 2·3, p. 80, and we have analogues of these propositions (Prop. 3·2, Prop. 4·1) we need only make changes in wording.

Finally we give conditions on a complex for the existence of a K-resolution. A simple extension X^* is said to admit *enough K-resolutions* if: (i) given any K-resolutions of X^0, X^2, there is a K-resolution M^{**} of X^* in which M^{0*}, M^{2*} are the given K-resolutions of X^0, X^2; (ii) whenever M^{**} is a normal resolution of X^* and M^{0*}, M^{2*} are K-resolutions of X^0, X^2, then M^{1*} is a K-resolution of X^1.

Proposition 4·3. *If X^* is a complex and all the simple extensions*

$$0 \to B^p(X) \to Z^p(X) \to H^p(X) \to 0, \quad 0 \to Z^p(X) \to X^p \to B'^p(X) \to 0$$

and their duals admit enough K-resolutions, then X^* admits a K-resolution.

Proof. The proof is essentially the proof of the first part of Prop. 1·2, p. 365 of Cartan & Eilenberg (1956). First choose K-resolutions of $B^p(X)$ and $H^p(X)$ for all p. Then

$$0 \to B^p(X) \to Z^p(X) \to H^p(X) \to 0$$

has a K-resolution containing the given K-resolutions. Since $B'^p(X) \cong B^{p+1}(X)$, we have a K-resolution of $B'^p(X)$, and we have also constructed a K-resolution of $Z^p(X)$. So there exists a K-resolution of $0 \to Z^p(X) \to X^p \to B'^p(X) \to 0$ containing these two resolutions. Let M^{p*} be the resolution of X^p determined by this resolution. Then the set of complexes $\{M^{p*}\}$ becomes a double complex M^{**} with δ_2 defined by the differentiations on the complexes M^{p*}, and δ_1 defined as the product of the morphisms which cover

$$X^p \to B'^p(X) \cong B^{p+1}(X) \to Z^{p+1}(X) \to X^{p+1}.$$

To verify that it is a K-resolution it is sufficient to show that it is normal, and that M^{p*}, $Z_1(M^{p*})$, etc., are K-resolutions of X^p, $Z(X^p)$, etc. Since δ_1 is defined as the product of an epimorphism and two monomorphisms which split, it follows that $\ker \delta_1$ and $\operatorname{coker} \delta_1$ split. So M^{**} is a normal resolution. By the construction it is immediate that all of M^{p*}, $Z_1(M^{p*})$, etc., except possible $Z_1'(M^{p*})$, are K-resolutions. Now

$$0 \to H_1(M^{p*}) \to Z_1'(M^{p*}) \to B_1'(M^{p*}) \to 0$$

is a normal resolution of $0 \to H^p(X) \to Z'^p(X) \to B'^p(X) \to 0$, since M^{**} is normal, and $H_1(M^{p*})$, $B'(M^{p*})$ are K-resolutions. So $Z_1'(M^{p*})$ is a K-resolution.

5. An exact couple relating resolutions of categories

First we recall the homology theory of filtered complexes of which full accounts are given by Cartan & Eilenberg (1956), Massey (1952), and Zeeman (1957).

A filtered $\mathfrak{C}$-complex $\{F, W\}$ is an object W of $\mathfrak{C}$, a family of subobjects $F^p(W)$, indexed by the integers, such that $F^p(W) \supset F^{p+1}(W)$, $F^p(W) = W (p \leqslant 0)$, and a differentiation d such that $dF^p(W) \subset F^p(W)$. If also W is graded, we shall assume that the filtration is compatible with the gradation and d is homogeneous of degree 1. We shall make the convention that $F^\infty(W)$ is the zero object, and write F^p for $F^p(W)$. Let H be the operation of forming the homology object with respect to d. Then the rth exact couple of $\{F, W\}$ is an exact sequence

$$\to C_r^{p+1} \overset{\alpha_r}{\to} C_r^p \overset{\beta_r^p}{\to} E_r^p \overset{\delta_r^p}{\to} C_r^{p+r} \to \quad (r \geqslant 1),$$

where: $C_r^p = \operatorname{Im}[H(F^p) \to H(F^{p-r+1})]$, $E_r^p = \operatorname{Im}[H(F^p/F^{p+r}) \to H(F^{p-r+1}/F^{p+1})]$

(these morphisms being induced by the inclusion $F^p \subset F^{p-r+1}$); α^p_r, β^p_r are induced by inclusions, and δ^p_r by d. The exact sequences may be recorded in the form

$$\begin{array}{ccc} C_r & \overset{\alpha_r}{\to} & C_r, \\ & {\scriptstyle \delta_r}\nwarrow \quad \swarrow{\scriptstyle \beta_r} & \\ & E_r & \end{array}$$

where E_r, C_r are the graded objects with components E^p_r, C^p_r, and α_r, β_r, δ_r have components α^p_r, β^p_r, δ^p_r. The morphism $d_r = \beta_r \delta_r$ is the differentiation of E_r induced by d, and $E_{r+1} = H(E_r)$. The sequence of complexes (E_r, d_r) is the spectral sequence of $\{F, W\}$. It is shown by Massey (1952) that the $(r+1)$th exact couple can be constructed from the rth exact couple. In particular C^{p+1}_{r+1} is $\operatorname{Im} \alpha^p_r$, and α^p_{r+1} is the restriction of α^{p-1}_r to C^{p+1}_{r+1}.

Let K and L be right resolutions of $\mathfrak{C}$ such that every K-resolution has an L-resolution. Let $A \in \mathfrak{C}$. Then A has a K-resolution X^*, and X^* has an L-resolution M^{**}. By propositions 3·1 and 4·2 M^{**} is invariant up to homotopy. We shall construct an invariant exact couple associated with TM^{**}, where T is a covariant functor on $\mathfrak{C}$ with values in an abelian category $\mathfrak{D}$.

We use the notation of § 4 and write δ for the total differentiation on M^{**} given by

$$\delta^{pq} = \delta^{pq}_1 + (-1)^p \delta^{pq}_2.$$

Let F_1, F_2 be the two filtrations on TM^{**} given by

$$F^p_1(TM^{**}) = \sum_{i \geq p} \sum_j TM^{ij}, \quad F^p_2(TM^{**}) = \sum_i \sum_{j \geq p} TM^{ij}.$$

Write d_1, d_2, and d for $T\delta_1$, $T\delta_2$, and $T\delta$. Regard TM^{**} as a complex with differentiation d and graded by the total degree. Then we have two filtered graded $\mathfrak{D}$-complexes $\{F_1, TM^{**}\}$, $\{F_2, TM^{**}\}$.

First we shall prove that the spectral sequence of $\{F_2, TM^{**}\}$ collapses to LTA. The E_2-term is obtained by first taking the homology with respect to d_1, and then with respect to the morphism induced by d_2. Since δ_1 splits, $H_1(TM^{**}) = TH_1(M^{**})$. But $H_1(M^{**})$ is by definition an L-resolution of $H(X^*)$, and $H(X^*)$ is a graded object whose only non-zero component is A in degree zero. Thus $H_1(M^{0*})$ is an L-resolution of A, and $H_1(M^{i*})$ is an L-resolution of the zero object for $i \neq 0$. So the homology of $TH_1(M^{**})$ with respect to the morphism induced by d_2 is LTA, where A is regarded as a graded object whose only non-zero component is A with degree zero. Thus the spectral sequence collapses and its limit is LTA. Hence the total homology of the complex TM^{**} is LTA.

In the remainder of this section we shall study the exact couple associated with $\{F_1, TM^{**}\}$. Since M^{**} is determined by A up to homotopy the second exact couple of $\{F_1, TM^{**}\}$ and its derived exact couples are determined by K, L, T, and A. We shall denote this sequence of exact couples by $(K, L)\ TA$. Propositions 3·1 and 4·2 show that a morphism α of A into B can be covered by a morphism α^{**} of M^{**} into an L-resolution N^{**} of a K-resolution Y^{**} of B, and α^{**} is determined up to homotopy. Write $(K, L)\ T(\alpha)$ for the morphism of $(K, L)\ TA$ into $(K, L)\ TB$ determined by α^{**}. Then it can be verified that $(K, L)\ T$ is a covariant functor. We call it the (K, L)-*exact couple of* T. Let K' and L' be right resolutions of $\mathfrak{C}$ dominating K and L respectively, and such that every K'-resolution has an L'-resolution. If Y'^* is a K'-resolution of B and N'^{**} is an L'-resolution of Y'^*,

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then α can be covered by a morphism of M^{**} into N'^{**} and such morphisms are determined up to homotopy. Thus we have determined a transformation $(K, L)\ T \to (K', L')\ T$, and it may be verified that this is a natural transformation of functors. We shall call it the τ-transformation of $(K, L)\ T$ into $(K', L')\ T$.

Now we calculate the E_2-terms of $(K, L)\ TA$. They are obtained by taking the homology of TM^{**} with respect to d_2, and then with respect to the morphism induced by d_1. Since M^{p*} is an L-resolution of X^p, $H_2^{pq}(TM^{**})$ is L^qTX^p. The morphism δ_1^{p*} is a morphism of M^{p*} into M^{p+1*} covering δ_X. So d_1 induces $L^qT(\delta_X)$ on the complex L^qTX^*. Hence the homology with respect to this morphism is K^pL^qTA, since X^* is a K-resolution of A. We have already seen that the total homology of TM^{**} is LTA, so we have proved:

THEOREM 5·1. *If K and L are right resolutions of $\mathfrak{C}$ such that every K-resolution has an L-resolution and T is a covariant functor on $\mathfrak{C}$ with values in an abelian category, the (K, L)-exact couple of T determines a spectral sequence*

$$K^pL^qT \underset{p}{\Rightarrow} L^nT.$$

Next we calculate the C_2-terms of $(K, L)\ T$. We shall denote them by C_2, or by $C_2(K, L)\ T$ when it is necessary to be more explicit. By definition

$$C_2^{pq}A = \operatorname{Im}[H^{p+q}(F_1^p\, TM^{**}) \to H^{p+q}(F_1^{p-1}\, TM)],$$

where H is the operation of taking homology with respect to d and the morphism is induced by inclusion. To calculate $C_2^{pq}A$ we need:

LEMMA 5·1. *Let N^{**} be a double complex with $N^{ij} = 0$ if i or $j < 0$, and differentiations d_1, d_2 of degrees $(1, 0)$ and $(0, 1)$ respectively. If the columns N^{*q} of N are acyclic, then for $p > 0$ there is a commutative diagram*

$$\begin{array}{ccc} H^{p+q}(F_1^p N^{**}) & \cong & H_2^q(\operatorname{Ker} d_1^{p*}) \\ \alpha\downarrow & & \beta\downarrow \\ H^{p+q}(F_1^{p-1} N^{**}) & \cong & H_2^{q+1}(\operatorname{Ker} d_1^{p-1*}) \end{array}$$

where: α is induced by inclusion; β is the connecting morphism determined by the exact sequence of complexes $0 \to \operatorname{Ker} d_1^{p-1*} \to N^{p-1*} \to \operatorname{Ker} d_1^{p*} \to 0$; H and H_2 are the operations of taking homology with respect to d, and the restriction of d_2 to $\operatorname{Ker} d_1^{p*}$.

Proof. Consider the complex N_p^{**} of N^{**} defined by $N_p^{ij} = N^{ij}$ $(i \geqslant p)$, $N_p^{ij} = 0$ $(i < p)$. Since the columns of N^{**} are acyclic so are those of N_p^{**}. Hence the spectral sequence associated with $\{F_2, N_p^{**}\}$ collapses, and we have

$$H(N_p^{**}) \cong H_2(\operatorname{Ker} d_1^{p*}).$$

Now we consider the subcomplex J^{**} of N^{**} with two non-zero rows N^{p-1*} and $\operatorname{Ker} d_1^{p*}$. Its columns are acyclic since $N^{p-1*} \to \operatorname{Ker} d_1^{p*}$ is an epimorphism. So the preceding argument applied to J^{**} instead of N_p^{**} shows that

$$H(J^{**}) \cong H_2(\operatorname{Ker} d_1^{p-1*}).$$

So we have an isomorphism $H(J^{**}) \cong H(N_{p-1}^{**})$ induced by the inclusion $J^{**} \subset N_{p-1}^{**}$.

The diagram

$$\begin{array}{ccc} \operatorname{Ker} d_1^{p*} & \to & N_p^{**} \\ \downarrow & & \downarrow \\ J^{**} & \to & N_{p-1}^{**}, \end{array}$$

with morphisms induced by inclusions, is commutative. So by taking the homology with respect to d, and noticing that d_2 and d induce the same morphism on $\operatorname{Ker} d_1^{p*}$, we have a commutative diagram

$$\begin{array}{ccc} H_2(\operatorname{Ker} d_1^{p*}) & \cong & H(N_p^{**}). \\ \downarrow & & \downarrow \\ H(J^{**}) & \cong & H(N_{p-1}^{**}). \end{array}$$

From Cartan & Eilenberg (1956, p. 332, case 3) the morphism

$$H_2(\operatorname{Ker} d_1^{p*}) \to H(J^{**}) \cong H_2(\operatorname{Ker} d_1^{p-1*})$$

is the connecting morphism in the exact homology sequence associated with the exact sequence of complexes $0 \to \operatorname{Ker} d_1^{p-1*} \to N^{p-1*} \to \operatorname{Ker} d_1^{p*} \to 0$. Since

$$H^{p+q}(F_1^p N^{**}) \cong H^{p+q}(N_p^{**}),$$

the lemma follows.

Suppose that $p > 0$ and apply the lemma to the preceding formula for $C_2^{pq} A$. Then we have

$$C_2^{pq} A = \operatorname{Im}[H_2^q(\operatorname{Ker} d_1^{p*}) \to H_2^{q+1}(\operatorname{Ker} d_1^{p-1*})],$$

where the morphism is the connecting morphism associated with the exact homology sequence of $0 \to \operatorname{Ker} d_1^{p-1*} \to TM^{p-1*} \to \operatorname{Ker} d_1^{p*} \to 0$. So from the exact homology sequence

$$C_2^{pq} A = \operatorname{Coker}[H_2^q(TM^{p-1*}) \to H_2^q(\operatorname{Ker} d_1^{p*})],$$

where the morphism is induced by d_1^{p-1*}. Since δ_1 splits and $d_1 = T(\delta_1)$, this is

$$C_2^{pq} A = \operatorname{Coker}[H_2^q(TM^{p-1*}) \to H_2^q(T \operatorname{Ker} \delta_1^{p*})].$$

But M^{p-1*} and $\operatorname{Ker} \delta_1^{p*}$ are L-resolutions of X^{p-1} and $\operatorname{Ker} \delta_X^p$. So $H_2^q(TM^{p-1*})$ and $H_2^q(T \operatorname{Ker} \delta_1^{p*})$ are $L^q TX^{p-1}$ and $L^q T \operatorname{Ker} \delta_X^p$. Thus

$$C_2^{pq} A = \operatorname{Coker} L^q T(\operatorname{coim} \delta_X^{p-1}).$$

But X^* is a K-resolution of A. So we have proved that

$$C_2^{pq} = \check{K}^p L^q T \quad (p > 0).$$

For $p \leqslant 0$, $F_1^p(TM^{**}) = TM^{**}$. So

$$C_2^{pq} A = H^{p+q}(TM^{**}) = L^{p+q} TA \quad (p \leqslant 0).$$

We summarize these results in:

Theorem 5·2. *The C_2^{pq}-term of the (K, L)-exact couple of T is $\check{K}^p L^q T$ for $p > 0$, and $L^{p+q} T$ for $p \leqslant 0$.*

Theorems 5·1 and 5·2 show that the first of the (K, L)-exact couples of T consists of sequences of natural transformations

$$\to \check{K}^{p+1} L^{q-1} T \to \check{K}^p L^q T \to K^p L^q T \to \check{K}^{p+2} L^{q-1} T \to$$

defined for each positive integer q. We denote the successive natural transformations by α_2^{pq}, β_2^{pq} and δ_2^{pq}. First we use the natural transformation α_2 to obtain a formula for the filtration of $L^n T$ determined by the (K, L)-exact couple. By definition C_j^i is the image of C_{j-1}^i in C_{j-1}^{i-1} under α_{j-1}^{i-1}. So C_r^p is the image of C_2^p in C_2^{p-r+2} under $\alpha_2^{p-r+2} \ldots \alpha_2^{p-2} \alpha_2^{p-1}$. By definition the component of L^nT with filtration p is $C_r^{p,n-p}$ for $r \geqslant p+1$. So by putting $r = p+2$ we have:

PROPOSITION 5·1. *The component of L^nT with filtration p in the filtration determined by the (K, L)-exact couple is the image of $\check{K}^pL^{n-p}T$ in L^nT under $\alpha_2^{0p}\alpha_2^{1p}\ldots\alpha_2^{p-1,1}$.*

We shall determine the transformation α_2 in a special case in §11. We conclude this section by determining β_2. By the definition of β_2^{pq} the diagram

$$\begin{array}{ccc} H^{p+q}(F_1^p\, TM^{**}) & \to & \check{K}^pL^qTA \\ \downarrow & & \downarrow \beta_2^{pq} \\ H^{p+q}(F_1^p\, TM^{**}/F_1^{p+1}\, TM^{**}) & \to & K^pL^qTA, \end{array}$$

where the left-hand morphism is induced by the canonical epimorphism and the rows are the defining epimorphisms for $C_2^{pq}A$, $E_2^{pq}A$, is commutative. By lemma 5·1 $H^{p+q}(F_1^p\, TM^{**})$ is $H_2^q(\operatorname{Ker} d_1^{p*})$. So the left-hand morphism is the morphism

$$H_2^q(\operatorname{Ker} d_1^{p*}) \to H_2^q(TM^{p*})$$

induced by $\ker d_1^{p*}$; and this is just the morphism

$$L^qT\operatorname{Ker}\delta_X^p \to L^qTX^p$$

induced by $\ker \delta_X^p$. So we have proved:

PROPOSITION 5·2. *The transformation $\beta_2^{pq}: \check{K}^pL^qT \to K^pL^qT$ is the natural transformation from the K-satellite to the K-derived functor.*

6. SHIFTING THEOREMS FOR FUNCTORS OF TWO VARIABLES

Let $\mathfrak{C}$ and $\mathfrak{C}'$ be abelian categories and T be a covariant functor from $\mathfrak{C}\times\mathfrak{C}'$ into an abelian category $\mathfrak{D}$. In later applications T will be $\operatorname{Hom}_{\mathfrak{C}}$ and $\mathfrak{C}$ will be the dual of $\mathfrak{C}'$, or T will be a tensor product. Further let K and K' be right resolutions of $\mathfrak{C}$ and $\mathfrak{C}'$. Then we write KT, $\check{K}T$ and $\hat{K}T$ for the K-derived functors, K-satellites and K-cosatellites of T obtained by regarding T as a class of functors defined on $\mathfrak{C}$ and indexed by $\mathfrak{C}'$. Similarly $K'T$, $\check{K}'T$ and $\hat{K}'T$ are obtained by regarding T as a class of functors defined on $\mathfrak{C}'$ and indexed by $\mathfrak{C}$.

Let X^* be a K-resolution of A, and X'^* be a K'-resolution of A'. Then the double complex $T(X^*, X'^*)$ is determined up to homotopy. Regard $T(X^*, X'^*)$ as a complex graded by the total degree and with differentiation the total differentiation. Let F_1, F_2 be the filtrations given by

$$F_1^p\, T(X^*, X'^*) = \sum_{i\geqslant p}\sum_j T(X^i, X'^j), \quad F_2^p\, T(X^*, X'^*) = \sum_i \sum_{j\geqslant p} T(X^i, X'^j).$$

Since $T(X^*, X'^*)$ is determined up to homotopy the second exact couple obtained from F_1, and its derived exact couples are invariant. Denote this sequence of exact couples by $(K * K')\, T(A, A')$. Similarly F_2 determines a sequence of exact couples which we denote by $(K' * K)\, T(A, A')$. It can be verified that $(K * K')\, T(A, A')$ and $(K' * K)\, T(A, A')$ are the values of functors $(K * K')\, T$ and $(K' * K)\, T$ on $\mathfrak{C}\times\mathfrak{C}'$. In particular the total homology of $T(X^*, X'^*)$ is given by a functor from $\mathfrak{C}\times\mathfrak{C}'$ to $\mathfrak{D}$. We shall denote it by both $(K\times K')\, T$ and $(K'\times K)\, T$.

The augmentations of the single complexes X^* and X'^* determine natural transformations

$$K'T \to (K\times K')\, T \leftarrow KT. \tag{6·1}$$

A theorem which gives criteria for these transformations to determine isomorphisms will be called a *shifting theorem*. More generally we shall apply this term to any theorem that enables us to replace m-fold and n-fold resolutions in $\mathfrak{C}$ and $\mathfrak{C}'$ by $(m-1)$-fold and $(n+1)$-fold resolutions in $\mathfrak{C}$ and $\mathfrak{C}'$. Such a theorem will be obtained by embedding the relevant complexes in an $(m+n)$-fold complex.

We shall need to know the E_2-terms of $(K * K')\, T$. They are obtained by taking first the homology with respect to the differentiation induced by $d_{X'}$, and then with respect to the differentiation induced by d_X. So the E_2^{pq}-term is $K^pK'^qT$, that is the K^p-derived functor of K'^qT. Thus we have shown that there is a spectral functor

$$K^pK'^qT \underset{p}{\Rightarrow} (K\times K')^n\, T \tag{6·2}$$

defined for each covariant functor T on $\mathfrak{C}\times\mathfrak{C}'$. Similarly we obtain from $(K' * K)\, T$ a spectral functor

$$K'^pK^qT \underset{p}{\Rightarrow} (K\times K')^n\, T. \tag{6·2'}$$

The morphism $KT(A,A') \to (K\times K')\, T(A,A')$ obtained from (6·1) can be factorized as

$$KT(A,A') \to KK'^0T(A,A') \to (K\times K')\, T(A,A'),$$

where the first morphism is the morphism obtained from the canonical transformation $T\to K'^0T$, and the second is an edge transformation of (6·2). The edge morphisms of a spectral sequence are isomorphisms if the spectral sequence collapses. So we have proved the first shifting theorem:

PROPOSITION 6·1. *If* $KT(A,A') \to KK'^0T(A,A')$ *and* $K'T(A,A') \to K'K^0T(A,A')$ *are isomorphisms, and* $K^pK'^qT(A,A')$ *and* $K'^pK^qT(A,A')$ *vanish for* $q>0$, *then the morphisms*

$$K'T(A,A') \to (K\times K')\, T(A,A') \leftarrow KT(A,A')$$

are isomorphisms.

Let L be a resolution of $\mathfrak{C}$ such that every K-resolution has an L-resolution. Define the (K,L)-*exact couple of* T to be the functor $(K,L)\, T$ given by regarding T as a class of functors on $\mathfrak{C}$ indexed by the objects of $\mathfrak{C}'$. Then a shifting theorem for $(K,L)\, T$ replaces a double complex in $\mathfrak{C}$ by a pair of single complexes one in $\mathfrak{C}$ and the other in $\mathfrak{C}'$. So we have two kinds of shifting theorems: the first kind replaces L-resolutions by K'-resolutions and gives criteria for $(K,L)\, T$ and $(K * K')\, T$ to be isomorphic; the second kind replaces K-resolutions by K'-resolutions and gives criteria for $(K,L)\, T$ and $(K' * L)\, T$ to be isomorphic.

Let X^* be a K-resolution of A, M^{**} be an L-resolution of X^*, and X'^* a K'-resolution of A'. Then the augmentations $A'\to X'^*$ and $X^*\to M^{**}$ induce morphisms

$$T(M^{**},A') \to T(M^{**},X'^*) \leftarrow T(X^*,X'^*) \tag{6·3}$$

which commute with the differentiations. To obtain the shifting theorems we shall give the triple complex $T(M^{**},X'^*)$ two different double complex structures. Write W^{***} for the triple complex $T(M^{**},X'^*)$ and d_1, d_2, d_3 for its differentiations; thus

$$W^{ijk} = T(M^{ij},X'^k),\quad d_1 = T(\delta_{M1},1),\quad d_2 = T(\delta_{M2},1),\quad d_3 = T(1,\delta_{X'}).$$

Let P^{**} be the double complex with components P^{rs} and differentiations δ_{P1}, δ_{P2} given by

$$P^{rs} = \sum_{j+k=s} W^{rjk},$$

$$\delta^{rs}_{P1} = \sum_{j+k=s} d_1^{rjk}, \quad \delta^{rs}_{P2} = \sum_{j+k=s} (d_3^{rjk} + (-1)^k d_2^{rjk}).$$

Then the morphisms (6·3) induce double complex morphisms

$$T(M^{**}, A') \rightarrow P^{**} \leftarrow T(X^*, X'^*). \tag{6·4}$$

Since the triple complex W^{***} is determined up to homotopy so is P^{**}. Hence the second exact couple obtained from the filtration on P^{**} given by the first index is determined by A and A'. Denote this exact couple and its derived exact couples by $(K, L\times K')\,T(A, A')$. It is easily verified that this is the value for the argument $A\times A'$ of a functor $(K, L\times K')\,T$ defined on $\mathfrak{C}\times\mathfrak{C}'$. Then (6·4) yield natural transformations of exact couple functors

$$(K, L)\,T \rightarrow (K, L\times K')\,T \leftarrow (K * K')\,T. \tag{6·5}$$

Next we obtain the E_2-terms of $(K, L\times K)\,T$. They are calculated by taking the homology of P^{**} first with respect to δ_{P2} and then with respect to the morphism induced by δ_{P1}. The complex P^{r*} is obtained from the double complex $T(M^{r*}, X'^*)$ by grading with the total degree and taking the total differentiation as differentiation. The homology of P^{r*} is $(L\times K')\,T(X^r, A')$, since M^{r*} and X'^* are respectively an L-resolution of X^r and a K'-resolution of A'. Since δ_{P1} may be identified with the morphism δ_{M1} which covers δ_X, the morphism induced by δ_{P1} on $(L\times K')\,T(X^*, A')$ is $(L\times K')\,T(\delta_X, 1)$. Since X^* is a K-resolution of A, the homology is $K(L\times K')\,T(A, A')$. Thus the E_2^{pq}-term of $(K, L\times K')\,T$ is $K^p(L\times K')^q\,T$.

Since the transformations (6·5) are induced by the augmentations of X'^* and M^{**}, it follows that the transformations of the E_2^{pq}-terms are induced by the canonical transformations of LT and $K'T$ into $(L\times K')\,T$. So we have proved:

Proposition 6·2. *The E_2^{pq}-term of $(K, L\times K')\,T$ is $K^p(L\times K')^q\,T$, and the restrictions*

$$K^pL^qT \rightarrow K^p(L\times K')^q\,T \leftarrow K^pK'^qT$$

of (6·5) *to the E_2^{pq}-terms are induced by the canonical transformations*

$$L^qT \rightarrow (L\times K')^q\,T \leftarrow K'^qT.$$

Now let Q^{**} be the double complex with components Q^{rs} and differentiations δ_{Q1}, δ_{Q2} given by

$$Q^{rs} = \sum_{i+k=r} W^{isk},$$

$$\delta^{rs}_{Q1} = \sum_{i+k=r} (d_3^{isk} + (-1)^k d_1^{isk}), \quad \delta^{rs}_{Q2} = \sum_{i+k=r} d_2^{isk}.$$

Write Y^* for $H_1^{0*}(M)$, i.e. $Y^q = \operatorname{Ker}\delta^{0q}_{M1}$, and Y^* has differentiation induced by δ_{M2}. Then the augmentations $A' \rightarrow X'^*$, and $Y^* \rightarrow M^{**}$ give morphisms of double complexes

$$T(M^{**}, A') \rightarrow Q^{**} \leftarrow T(Y^*, X'^*).$$

The complex Q^{**} is determined up to homotopy, and by the usual arguments we can prove that the second exact couple and its derived exact couples associated with the filtration on Q^{**} defined by the first index gives a covariant functor on $\mathfrak{C}\times\mathfrak{C}'$. Denote it by $(K\times K', L)\,T$. Then (6·3) gives morphisms of exact couples

$$(K, L)\,T \to (K\times K', L)\,T \leftarrow (K' * L)\,T. \tag{6·6}$$

The E_2-terms of $(K\times K', L)\,T$ are obtained by taking the homology of Q^{**} with respect to δ_{Q2} and then with respect to the morphism induced by δ_{Q1}. The homology of Q^{r*} with respect to δ_{Q2} is $\sum_{i+k=r} LT(X^i, X'^k)$, since M^{i*} is an L-resolution of X^i. The morphism δ_{Q1} induces the total differentiation of $LT(X^*, X'^*)$. So the homology with respect to the morphism induced by δ_{Q1} is $(K\times K')\,LT(A, A')$, for X^* is a K-resolution of A and X'^* is a K'-resolution of A'. Hence the E_2^{pq}-term of $(K\times K', L)\,T$ is $(K\times K')^p\,L^q T$.

Since the transformations (6·6) are induced by the augmentations of X' and M^{**} it follows that the transformations of the E_2^{pq}-terms are induced by the canonical transformations of KT and $K'T$ into $(K\times K')\,T$. So we have proved:

Proposition 6·3. *The E_2^{pq}-term of $(K\times K', L)\,T$ is $(K\times K')^p\,L^q T$ and the restrictions*

$$K^p L^q T \to (K\times K')^p\,L^q T \leftarrow K'^p L^q T$$

of (6·6) *to the E_2^{pq}-terms are the canonical transformations* (6·1) *with T replaced by $L^q T$.*

In order to deduce shifting theorems from propositions 6·2 and 6·3 we need a criterion for a morphism of exact couples to be an isomorphism:

Lemma 6·1. *Let $\{C_r^{pq}, E_r^{pq}\}$ and $\{D_r^{pq}, F_r^{pq}\}$ be the exact couples associated with the filtrations on the first indices of double complexes Y^{**} and Z^{**}, respectively. If $Y^{pq} = Z^{pq} = 0$ for $p, q < 0$, then a complex morphism f of Y^{**} into Z^{**} determines isomorphisms of $\{C_r^{pq}, E_r^{pq}\}$ on to $\{D_r^{pq}, F_r^{pq}\}$ for $r \geqslant 2$ if it determines isomorphisms of E_2 into F_2.*

Proof. Since C_r $(r>2)$ is a functor of C_2 it is sufficient to show that f determines an isomorphism on C_2. Since f induces an isomorphism of E_2 onto F_2 it induces an isomorphism of the spectral sequences, in particular an isomorphism of C_2^{0q} (the total homology of Y^{**}) onto D_2^{0q} (the total homology of Z^{**}). Suppose that the induced morphisms of C_2^{pq} into D_2^{pq} are isomorphisms for $p<n$. Then the five-lemma applied to

$$\begin{array}{ccccccccc} C_2^{n-2,\,q+1} & \to & E_2^{n-2,\,q+1} & \to & C_2^{nq} & \to & C_2^{n-1,\,q} & \to & E_2^{n-1,\,q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D_2^{n-2,\,q+1} & \to & F_2^{n-2,\,q+1} & \to & D_2^{nq} & \to & D_2^{n-1,\,q} & \to & F_2^{n-1,\,q} \end{array}$$

shows that the morphism of C_2^{nq} into D_2^{nq} is an isomorphism. So the lemma follows by induction on n.

Write $|K|$ for the class of objects that are components of K-resolutions. Then the first shifting theorem for $(K, L)\,T$ is:

Theorem 6·1. *The exact couples $(K, L)\,T$ and $(K * K')\,T$ are isomorphic if the canonical morphisms*

$$LT(A, A') \to (L\times K')\,T(A, A') \leftarrow K'T(A, A')$$

are isomorphisms for A in $|K|$.

Proof. By the hypothesis the transformations

$$KLT \to K(L\times K')\,T \leftarrow KK'T$$

are isomorphisms. Hence (6·5), proposition 6·2, and lemma 6·1 show that $(K, L)\ T$ and $(K * K')\ T$ are isomorphic.

Corollary. *If the canonical morphisms*

$$LT(A, A') \to LK'^0 T(A, A'), \quad K'T(A, A') \to K'L^0 T(A, A')$$

are isomorphisms for A in $|K|$, *and* $L^p K'^q T(A, A')$, $K'^p L^q T(A, A')$ *vanish for* $q > 0$ *and A in* $|K|$, *then* $(K, L)\ T$ *and* $(K * K')\ T$ *are isomorphic.*

Proof. Proposition 6·1 shows that the conditions of the theorem are satisfied.

The second shifting theorem is:

Theorem 6·2. *The exact couples* $(K, L)\ T$ *and* $(K' * L)\ T$ *are isomorphic if the canonical transformations*

$$K'LT \to (K \times K')\ LT \leftarrow KLT$$

are isomorphisms.

Proof. It follows from (6·6), proposition 6·3, and lemma 6·1 that $(K, L)\ T$ and $(K' * L)\ T$ are isomorphic.

Corollary. *If the canonical transformations*

$$K'LT \to K'K^0 LT, \quad KLT \to KK'^0 LT$$

are isomorphisms, and $K^p K'^q LT$, $K'^p K^q LT$ *vanish for* $q > 0$, *then* $(K, L)\ T$ *and* $(K' * L)\ T$ *are isomorphic.*

Proof. By proposition 6·1 the condition of the theorem is satisfied, so the corollary follows.

7. A class of resolutions defined by an E-functor

Most of the results of this section have been obtained implicitly by Buchsbaum (1959) or explicitly by Heller (1958). The construction of the resolutions requires the existence of enough objects to act as 'injectives' relative to the E-functor. First we shall obtain some simple facts about such objects, and then prove an existence theorem for resolutions of complexes.

Let Θ be an E-functor on an abelian category $\mathfrak{C}$. An object M of $\mathfrak{C}$ is said to be Θ-*injective* if for each simple extension A^* in $\tilde{\Theta}$ the sequence Hom (A^*, M) is exact. In the terminology of § 3 this means that M is injective over all Θ-monomorphisms. When Θ is Ext^1, we abbreviate Θ-injective to injective. It can be verified in the usual way that the direct product of a set of objects is Θ-injective if and only if the factors are Θ-injective.

Proposition 7·1. *An object M of* $\mathfrak{C}$ *is* Θ-*injective if and only if* $\Theta(X, M) = 0$ *for each X in* $\mathfrak{C}$.

Proof. If M is Θ-injective and $0 \to M \to A \to X \to 0$ represents an element x of $\Theta(X, M)$, then Hom $(A, M) \to$ Hom (M, M) is an epimorphism. So $M \to A$ has a left inverse. Hence $x = 0$, and $\Theta(X, M) = 0$.

Conversely suppose $\Theta(X, M) = 0$ for each X in $\mathfrak{C}$, and let $X^* \in \tilde{\Theta}$. Since

$$0 \to \mathrm{Hom}\,(X^2, M) \to \mathrm{Hom}\,(X^1, M) \to \mathrm{Hom}\,(X^0, M) \to \Theta(X^2, M)$$

is exact, Hom (X^*, M) is exact. So M is Θ-injective.

A simple Θ-extension $0 \to A \to M \to B \to 0$ will be called a Θ-injective representation of A if M is Θ-injective. If each object of $\mathfrak{C}$ has a Θ-injective representation, we shall say that $\mathfrak{C}$ *has sufficient* Θ*-injectives*, or that Θ *has sufficient injectives*. In this paper we are mainly concerned with E-functors for which there exist sufficient injectives (or projectives). The next theorem shows that in this case we can restrict ourselves to closed E-functors (in § 23 we construct a closed E-functor without sufficient projectives or injectives on the category of abelian groups).

THEOREM 7·1. *If* Θ *is an E-functor on* $\mathfrak{C}$ *with sufficient injectives, then* Θ *is closed.*

Proof. By theorem 1·1, to show that Θ is right-closed it is sufficient to show that $\beta\alpha$ is a Θ-morphism if α, β are both Θ-monomorphisms. Let α in Hom (A, B) and β in Hom (B, C) be Θ-monomorphisms, and μ be a Θ-monomorphism of A into a Θ-injective M. Then there exist morphisms ρ, σ such that $\rho\alpha = \mu$, and $\sigma\beta = \rho$. Hence $\sigma\beta\alpha = \mu$. Now μ is a Θ-monomorphism and $\beta\alpha$ is a monomorphism, so $\beta\alpha$ is a Θ-monomorphism by axiom (d) for f. classes.

Lastly we show that Θ is left-closed. Let X^* be an element of $\tilde{\Theta}$, and $X \in \mathfrak{C}$. Choose a Θ-injective representation $0 \to X \to M \to Y \to 0$ of X. Since $\Theta(\;, M) = 0$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}\,(X^2, M) & \to & \mathrm{Hom}\,(X^1, M) & \to & \mathrm{Hom}\,(X^0, M) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathrm{Hom}\,(X^2, Y) & \to & \mathrm{Hom}\,(X^1, Y) & \to & \mathrm{Hom}\,(X^0, Y) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & \Theta(X^2, X) & \to & \Theta(X^1, X) & \to & \Theta(X^0, X) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

in which the columns and first two rows are exact. The exactness of the bottom row is a consequence of these facts.

Later we shall need the following criterion—due to Heller (1958)—for a simple extension to be in $\tilde{\Theta}$.

PROPOSITION 7·2. *If* $\mathfrak{C}$ *has sufficient* Θ*-injectives, then a simple extension* X^* *is in* $\tilde{\Theta}$ *if and only if* Hom (X^*, M) *is exact for all* Θ*-injectives* M.

Proof. By the definition of Θ-injectives Hom (X^*, M) is exact when M is a Θ-injective and X^* is in $\tilde{\Theta}$. Conversely suppose that Hom (X^*, M) is exact whenever M is a Θ-injective. Let μ be a Θ-monomorphism of X^0 into a Θ-injective M. Then the exactness of Hom (X^*, M) shows that δ^0_X is a right factor of μ. So axiom (d) for f. classes shows that δ^0_X is a Θ-morphism. Hence X^* is in $\tilde{\Theta}$.

In the remaining part of this section we shall assume that Θ is an E-functor with sufficient injectives, and we shall discuss the properties of a class of right resolutions defined by Θ. A complex is called a Θ-*complex* if its differentiation is a Θ-morphism, and a Θ-*injective complex* if each object is a Θ-injective. An acyclic right Θ-injective Θ-complex with homology object A is called a Θ-*injective resolution of* A. It follows that the augmentation monomorphism of a Θ-injective resolution is a Θ-morphism. Since every object of $\mathfrak{C}$ has a Θ-injective representation the usual iteration argument shows that every object has a Θ-injective resolution.

If X^* and Y^* are two Θ-injective resolutions, then Y^n is injective over $\operatorname{im}\delta_X^n$ and $\ker\delta_X^n$ since they are Θ-monomorphisms. Further the zero resolution is Θ-injective. Hence the class of Θ-injective resolutions is a resolution of $\mathfrak{C}$. We shall denote this class by K_θ.

First we shall obtain some simple properties of $K_\theta T$, $\check{K}_\theta T$, and $\hat{K}_\theta T$. In particular we shall show that the study of the K_θ-derived functors and K_θ-cosatellites can be reduced to the study of the operations K_θ^0, $\hat{K}_\theta^0$, and $\check{K}_\theta^1$. If M is a Θ-injective, then $\ldots \to 0 \to M \to 0 \to \ldots$ is a K_θ-resolution of M. So $K_\theta^n TM$, $\check{K}_\theta^n TM$ vanish for $n \neq 0$, equal TM for $n = 0$, and $\hat{K}_\theta^n TM$ vanishes for all n. Let A be an object of $\mathfrak{C}$, and M^* be a K_θ-resolution of A. Then

$$\hat{K}_\theta^0 TM^* = 0 \quad \text{and} \quad K_\theta^0 TM^* \cong TM^*.$$

So for all n

$$K_\theta^n \hat{K}_\theta^0 T = 0 \quad \text{and} \quad K_\theta^n K_\theta^0 T \cong K_\theta^n T. \tag{7·1}$$

The first of these and the exactness property of the K_θ-sequence of $\hat{K}_\theta^0 T$ show that we have a natural isomorphism

$$\hat{K}_\theta^n \hat{K}_\theta^0 T \cong \check{K}_\theta^n \hat{K}_\theta^0 T. \tag{7·2}$$

Now we show that the natural transformation $\hat{K}_\theta^0 T \to T$ induces an isomorphism

$$\hat{K}_\theta^n \hat{K}_\theta^0 T \cong \hat{K}_\theta^n T. \tag{7·3}$$

Let A^n be $\operatorname{Ker}\delta_M^n$. Then

$$\hat{K}_\theta^n(\hat{K}_\theta^0 T)\, A \cong \operatorname{Ker}\{\hat{K}_\theta^0 TA^n \to \hat{K}_\theta^0 TM^n\} \cong \hat{K}_\theta^0 TA^n$$

since $\hat{K}_\theta^0 T$ vanishes on injectives, and

$$\hat{K}_\theta^n TA \cong \operatorname{Ker}\{TA^n \to TM^n\} \cong \hat{K}_\theta^0 TA^n$$

since $\ldots 0 \to M^n \to M^{n+1} \to \ldots$ is a K_θ-resolution of A^n. So we have proved that (7·3) is true. From (7·2) and (7·3) we deduce that $\hat{K}_\theta^n T$ is naturally isomorphic to $\check{K}_\theta^n \hat{K}_\theta^0 T$. So the study of $\hat{K}_\theta$ is reduced to the study of $\hat{K}_\theta^0$ and $\check{K}_\theta$. Since $\hat{K}_\theta^0 T$ is $\operatorname{Ker}[TA \to TM^0]$, $\hat{K}_\theta^0 T$ and T are isomorphic if T vanishes on Θ-injectives. In particular

$$\hat{K}_\theta^0 \check{K}_\theta^n T \cong \check{K}_\theta^n T \quad (n > 0). \tag{7·4}$$

To complete the description of the combinations of $\hat{K}_\theta^0$ and the other operations it is sufficient to show that

$$\hat{K}_\theta^0 K_\theta^0 T = 0. \tag{7·5}$$

We have $K_\theta^0 TA \cong \operatorname{Ker}[TM^0 \to TM^1]$, and $K_\theta^0 TM^0 \cong TM^0$. So the morphism $K_\theta^0 TA \to K_\theta^0 TM^0$ is the monomorphism $\ker[TM^0 \to TM^1]$. Hence $\hat{K}_\theta^0 K_\theta^0 T = 0$.

Next we obtain results for K_θ^n analogous to those of Cartan & Eilenberg (1956, chap. V, §§ 5, 6, 7). By (7·3) and (7·5) the K_θ-sequence for $K_\theta^0 T$ is

$$\to 0 \to \check{K}_\theta^n K_\theta^0 T \to K_\theta^n K_\theta^0 T \to 0 \to.$$

To show that this is exact it is sufficient to show that the K-excess of $K_\theta^0 T$ vanishes. By formula (3·3)

$$EK_\theta^n K_\theta^0 T \cong \operatorname{Ker}[K_\theta^0 TM^n \to K_\theta^0 TA^{n+1}]/\operatorname{Im}[K_\theta^0 TA^n \to K_\theta^0 TM^n].$$

But $K_\theta^0 TM^n \cong TM^n$, and $K_\theta^0 TA^{n+1} \cong \operatorname{Ker}[TM^{n+1} \to TM^{n+2}]$. Hence

$$\begin{aligned} EK_\theta^n K_\theta^0 T &\cong \operatorname{Ker}[TM^n \to \operatorname{Ker}[TM^{n+1} \to TM^{n+2}]]/\operatorname{Im}[\operatorname{Ker}[TM^n \to TM^{n+1}] \to TM^n] \\ &\cong \operatorname{Ker}[TM^n \to TM^{n+1}]/\operatorname{Ker}[TM^n \to TM^{n+1}] = 0. \end{aligned} \tag{7·7}$$

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So the K_θ-sequence is exact, and we have a natural isomorphism

$$\check{K}_\theta^n K_\theta^0 T \cong K_\theta^n K_\theta^0 T. \tag{7·8}$$

From (7·1) and (7·8) it follows that $K_\theta^n T$ is naturally isomorphic to $\check{K}_\theta^n K_\theta^0 T$. So the study of K_θ is reduced to the study of $\check{K}_\theta^n$ and K_θ^0. Since $K_\theta^0 TA$ is a component of the homology of TM^*, it vanishes if T vanishes on Θ-injectives. In particular

$$K_\theta^0 \check{K}_\theta^n T = 0 \quad (n > 0). \tag{7·9}$$

Finally we show that $\check{K}_\theta^n$ is obtained by iterating $\check{K}_\theta^1$. We have for $n > 0$

$$\begin{aligned} \check{K}_\theta^n(\check{K}_\theta^1 TA) &\cong \operatorname{Coker}[\check{K}_\theta^1 TM^{n-1} \to \check{K}_\theta^1 TA^n] \\ &\cong \check{K}_\theta^1 TA^n, \quad \text{since } \check{K}_\theta^1 T \text{ vanishes on } \Theta\text{-injectives,} \\ &\cong \operatorname{Coker}[TM^n \to TA^{n+1}] \cong \check{K}_\theta^{n+1} TA. \end{aligned}$$

Write θ for $\check{K}_\theta^1$, and define θ^n by $\theta^n T \cong \theta^{n-1}(\theta T)$, $\theta^0 T = T$. Then this shows together with the previous results for K_θ^n and $\hat{K}_\theta^n$ that there are natural isomorphisms

$$\theta^n T \cong \check{K}_\theta^n T, \quad \theta^n \hat{K}_\theta^0 T \cong \hat{K}_\theta^n T, \quad \theta^n K_\theta^0 T \cong K_\theta^n T. \tag{7·10}$$

We conclude this section by proving that every Θ-complex has a K_θ-resolution. The properties of an f. class of morphisms show that a complex X^* is a Θ-complex if and only if the exact sequences

$$0 \to B^p(X) \to Z^p(X) \to H^p(X) \to 0, \quad 0 \to Z^p(X) \to X^p \to B'^p(X) \to 0$$

and their duals are in $\tilde{\Theta}$ for all p. So by proposition 4·3 it is sufficient to prove that every simple Θ-extension has enough K_θ-resolutions. To prove this we need some properties of the exact and commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^0 & \overset{\mu^0}{\to} & M^0 & \to & B^0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^1 & \overset{\mu^1}{\to} & M^1 & \to & B^1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A^2 & \overset{\mu^2}{\to} & M^2 & \to & B^2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{\textbf{R}}$$

Denote the columns of **R** by A^*, M^*, B^* and the rows by r_0, r_1, r_2. We shall call **R** a Θ-*representation* of A^* if all its morphisms are Θ-morphisms, and a Θ-*injective representation* if, additionally, M^0, M^1 and M^2 are Θ-injectives.

LEMMA 7·1. (i) *If A^* is in $\tilde{\Theta}$, then there exists a diagram* **R** *in which r_2 is a given element of $\tilde{\Theta}(B^2, A^2)$, r_0 is a given Θ-injective representation of A^0, and M^* splits.*

(ii) *If A^*, M^*, r_0, and r_2 are in $\tilde{\Theta}$, then* **R** *is a Θ-representation of A^*.*

Proof. First we prove (i). Define M^1 to be $M^0 \oplus M^2$, δ_M^0 to be the canonical monomorphism of M^0 into M^1, and δ_M^1 to be the canonical epimorphism of M^1 onto M^2. Since M^0 is a Θ-injective and δ_A^0 is a Θ-monomorphism there exists a morphism μ' in Hom (A^1, M^0) such

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that $\mu'\delta_A^0 = \mu^0$. Put $\mu^1 = \mu' \oplus \mu^2\delta_A^1$. Then μ^1 is a monomorphism, for μ^0, μ^2 are monomorphisms and $\operatorname{Im}\delta_A^0 = \operatorname{Ker}\delta_A^1$. Define B^1 to be $\operatorname{Coker}\mu^1$, and the morphisms δ_B^0, δ_B^1 to be those induced by δ_M^0, δ_M^1. Thus we have constructed a diagram **R** with the required properties.

Now we prove (ii). We have $\delta_B^1 \operatorname{coker}\mu^1 = (\operatorname{coker}\mu^2)\,\delta_M^1$. By axiom (e_2) for h.f. classes $(\operatorname{coker}\mu^2)\,\delta_M^1$ is a Θ-morphism. Hence by axiom (d), δ_B^1 is a Θ-morphism. So B^* is in $\tilde{\Theta}$. To show that r_1 is in $\tilde{\Theta}$ we use proposition 7·2. Let M be a Θ-injective, and consider $\operatorname{Hom}(\mathbf{R}, M)$. By the definition of a Θ-injective the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Hom}(M^0, M) & \to & \operatorname{Hom}(M^1, M) & \to & \operatorname{Hom}(M^2, M) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \operatorname{Hom}(A^0, M) & \to & \operatorname{Hom}(A^1, M) & \to & \operatorname{Hom}(A^2, M) & \to 0 \end{array}$$

has exact rows and the first and last columns are epimorphisms. So the middle column is an epimorphism, and proposition 7·2 shows that r_1 is in $\tilde{\Theta}$. This completes the proof of the lemma.

Now we prove that any simple extension A^* in $\tilde{\Theta}$ has enough K_θ-resolutions. Let M^{**} be a resolution of A^*, write δ for the differentiation of M^{**} of type $(0, 1)$, and δ^i for its component of degree i. Put $A^{*i} = \operatorname{Ker}\delta^i$, and $B^{*i} = \operatorname{Coim}\delta^i$. Evidently $A^{*i} = B^{*i-1}$ $(i \geq 1)$, and $A^{*0} = A^*$. We shall write $\mathbf{R}_i$ for the commutative diagram

$$0 \to A^{*i} \to M^{*i} \to B^{*i} \to 0.$$

Then M^{**} is a K_θ-resolution of A^* if and only if $\mathbf{R}_i$ is a Θ-injective representation of A^{*i} for all $i \geq 0$. Suppose now that M^{0*}, M^{2*} are given K_θ-resolutions of A^0, A^2. We shall construct a K_θ-resolution M^{**} with these two rows. Lemma 7·1 (i) shows that we can construct a diagram $\mathbf{R}_0$ with the two given rows and a middle column that splits. Since M^{10} is the direct product of the Θ-injectives M^{00}, M^{20}, Lemma 7·1 (ii) shows that $\mathbf{R}_0$ is a Θ-injective representation of A^*. Since B^{*0} is in $\tilde{\Theta}$ we may repeat this process using B^{*0} instead of A^*, and so construct a Θ-injective representation $\mathbf{R}_1$ of B^{*0}. Continuing in this way we obtain a K_θ-resolution of A^* with the preassigned rows M^{0*}, M^{2*}. Next suppose that M^{**} is a normal resolution of A^*, and M^{0*}, M^{2*} are K_θ-resolutions of A^0, A^2. Since M^{*i} splits and M^{0i}, M^{2i} are Θ-injectives M^{1i} is a Θ-injective. Then lemma 7·1 (ii) shows that $\mathbf{R}_0$ is a Θ-injective representation of A^*. In particular B^{*0} is in $\tilde{\Theta}$; that is, A^{*1} is in $\tilde{\Theta}$. Again lemma 7·1 (ii) shows that $\mathbf{R}_1$ is a Θ-injective representation of A^{*1}. Continuing in this way it follows that $\mathbf{R}_i$ is a Θ-injective representation of A^{*i} for each i. So M^{**} is a K_θ-resolution of A. Thus we have proved that A^* has enough K_θ-resolutions, and from proposition 4·3 we deduce:

Proposition 7·3. *Every Θ-complex has a K_θ-resolution.*

In particular every member of $\tilde{\Theta}$ has a K_θ-resolution. So if $A^* \in \tilde{\Theta}$, we have an exact sequence

$$\to K_\theta^r TA^0 \to K_\theta^r TA^1 \to K_\theta^r TA^2 \to K_\theta^{r+1} TA^0 \to.$$

To describe this situation we extend the usual definition of connected sequences of functors, and we call a sequence of functors $\{T^i\}_{i \geq 0}$ Θ-*connected* if for each A^* in $\tilde{\Theta}$ there is a natural morphism $T^iA^2 \to T^{i+1}A^0$, and the sequence

$$0 \to T^0A^0 \to T^0A^1 \to \ldots \to T^nA^0 \to T^nA^1 \to T^nA^2 \to T^{n+1}A^0 \to$$

has order two. The existence of the natural morphism for each A^* in $\tilde{\Theta}$ may also be expressed by saying that there is a natural mapping of $\Theta(A^2, A^0)$ into $\mathrm{Hom}\,(T^iA^2, T^{i+1}A^0)$ for each $i \geqslant 0$. If the sequence is exact, then $\{T^i\}$ is called a *cohomological* Θ*-connected sequence*. In particular $K_\theta T$ is a cohomological Θ-connected sequence. Buchsbaum (1959) has shown that $\{\Theta^n(A, \ )\}$, where $\Theta^0 = \mathrm{Hom}$, is a cohomological Θ-connected sequence. To show that it is isomorphic to $K_\theta \mathrm{Hom}\,(A, \ )$ we need the following result whose proof is the proof of Prop. 2·2·1 of Grothendieck (1957) with only verbal modifications:

Proposition 7·4. *If $\{T^i\}$ is a cohomological Θ-connected sequence, $\{U^i\}$ is a Θ-connected sequence, T^i vanishes on Θ-injectives for $i > 0$, and $\mathfrak{C}$ has sufficient Θ-injectives, then a natural transformation of functors ρ^0: $T^0 \to U^0$ has a unique extension to a natural transformation ρ of Θ-connected sequences. Furthermore, if U^i vanishes on Θ-injectives for $i > 0$, and ρ^0 is an isomorphism, then ρ is an isomorphism.*

Since $\mathrm{Hom}\,(A, \ )$ is left-exact $K^0_\theta \mathrm{Hom}\,(A, \ ) \cong \Theta^0(A, \ )$. If M is Θ-injective, then $K^n_\theta \mathrm{Hom}\,(A, M) = 0$ for $n > 0$, and $\Theta^n(A, M) = 0$ since a simple Θ-extension beginning with a Θ-injective splits. So the proposition shows that $K_\theta \mathrm{Hom}$ and Θ are isomorphic.

8. Classes of resolutions associated with a pair of E-functors

Let Θ, Φ be a pair of E-functors with sufficient injectives such that $\Phi \subset \Theta$. Since Θ-injectives are Φ-injectives and Φ-morphisms are Θ-morphisms $K_\theta \succ K_\phi$. We shall construct a sequence of intermediate classes of resolutions.

Let k be a non-negative integer or ∞. We call a right resolution M^* a (Θ, k, Φ)*-injective resolution* if: for $n < k$, M^n is a Θ-injective and δ^n_M is a Θ-morphism; for $n \geqslant k$, M^n is a Φ-injective and δ^n_M is a Φ-morphism. In particular a $(\Theta, 0, \Phi)$-injective resolution is a K_ϕ-resolution and a (Θ, ∞, Φ)-resolution is a K_θ-resolution. Suppose that $k \leqslant l$, and let M^* be a (Θ, k, Φ)-resolution, N^* be a (Θ, l, Φ)-resolution. If $n < l$, then N^n is a Θ-injective and $\ker \delta^n_M$, $\mathrm{im}\, \delta^n_M$ are Θ-morphisms; if $n \geqslant l$, then N^n is a Φ-injective and $\ker \delta^n_M$, $\mathrm{im}\, \delta^n_M$ are Φ-morphisms. Hence for all n, N^n is injective over $\ker \delta^n_M$ and $\mathrm{im}\, \delta^n_M$. For $k = l$ this shows that the (Θ, k, Φ)-resolutions form a class of resolutions of $\mathfrak{C}$. Denote this class by K_k. Then for $k \leqslant l$ the remark shows that $K_k \prec K_l$.

Next we obtain some formulae relating $K^n_k T$, $\check{K}^n_k T$ and $\hat{K}^n_k T$. It is clear that

$$\check{K}^n_k T \cong \check{K}^n_\theta T \ \ (n \leqslant k), \quad \hat{K}^n_k T \cong \hat{K}^n_\theta T \ \ (n \leqslant k-1), \quad K^n_k T \cong K^n_\theta T \ \ (n < k-1). \qquad (8·1)$$

Let M^* be a K_k-resolution of A and $A^k = \mathrm{Ker}\, \delta^k_M$. Then

$$0 \to M^k \to M^{k+1} \to \dots$$

is a K_ϕ-resolution of A^k.

First suppose that $n > k$, and consider $\check{K}^n_k T$. Then $\check{K}^n_k TA = \phi^{n-k} TA^k$. But $\phi^{n-k} T$ vanishes on Φ-injectives, and M^{k-1} is a Φ-injective since $\Phi \subset \Theta$. So

$$\check{K}^n_k TA \cong \phi^{n-k} TA^k \cong \mathrm{Coker}\,[\phi^{n-k} TM^{k-1} \to \phi^{n-k} TA^k] \cong \theta^k \phi^{n-k} TA.$$

Thus we have an iteration formula

$$\check{K}^n_k T = \theta^k \phi^{n-k} T \quad (n > k). \qquad (8·2)$$

Secondly, suppose that $n \geqslant k$, and consider $\hat{K}^n_k T$. Then $\hat{K}^n_k TA = \hat{K}^{n-k}_\phi TA^k$. Again $\hat{K}^{n-k}_\phi T$ vanishes on Φ-injectives, and a similar argument shows that

$$\hat{K}^n_k T = \theta^k \hat{K}^{n-k}_\phi T = \theta^k \phi^{n-k} \hat{K}^0_\phi T \quad (n \geqslant k). \qquad (8·3)$$

Finally consider $K_k^n T$. By (8·3) and (7·5) the K_k-sequence for $K_\phi^0 T$ is for $n \geqslant k$

$$\rightarrow 0 \rightarrow \check{K}_k^n K_\phi^0 T \rightarrow K_k^n K_\phi^0 T \rightarrow 0 \rightarrow.$$

Since $0 \rightarrow M^k \rightarrow M^{k+1} \rightarrow \ldots$ is a K_ϕ-resolution of A^k

$$(EK_k)^n K_\phi^0 TA = (EK_\phi)^{n-k} K_\phi^0 TA^k.$$

So from (7·7) $(EK_k)^n K_\phi^0 T$ vanishes. Hence the K_k-sequence is exact for $n \geqslant k$, and we have

$$\check{K}_k^n K_\phi^0 T = K_k^n K_\phi^0 T \quad (n \geqslant k). \tag{8·4}$$

Now we have seen in §7 that $K_\phi^0 TM = TM$ when M is a Φ-injective. Therefore

$$K_\phi^0 TM^* = TM^*.$$

So $\check{K}_k^n K_\phi^0 T$ and $K_k^n T$ are isomorphic. Hence from (8·4) and (8·2)

$$K_k^n T = \theta^k \phi^{n-k} K_\phi^0 T \quad (n \geqslant k). \tag{8·5}$$

Thus except for $K_k^{k-1} T$ we have expressed $\check{K}_k T$, $\hat{K}_k T$ and $K_k T$ in terms of θ, ϕ, $\hat{K}_\theta^0$, $\hat{K}_\phi^0$, K_θ^0 and K_ϕ^0.

We also mention an interpretation of $K_k^n T$ when T is Hom $(X, \ )$. In this case $K_k^n TA$ is the group of equivalence classes of n-fold extensions

$$0 \rightarrow A \overset{\alpha_0}{\rightarrow} X_1 \overset{\alpha_1}{\rightarrow} X_1 \rightarrow \ldots \xrightarrow{\alpha_{n-1}} X_n \overset{\alpha_n}{\rightarrow} X \rightarrow 0,$$

where α_i $(i < k)$ are Θ-morphisms and α_i $(i \geqslant k)$ are Φ-morphisms. Since we shall not use this result, we only remark that the proof is an adaptation of Yoneda's proof of the analogous result for Ext^n.

Finally, we make a remark about the relationship between resolutions of categories, and resolutions defined by a nest of E-functors.

For a resolution K of an Abelian category $\mathfrak{C}$, denote by $|K|_n$ the class of objects of the form N^k, where N^* is in K and $0 \leqslant k \leqslant n$. In §15 we shall prove that the simple extensions A^* such that Hom (A^*, X) is exact for each object X of an arbitrary subclass of $\mathfrak{C}$ are the representatives of a closed E-functor on $\mathfrak{C}$. Let Θ_n be the E-functor on $\mathfrak{C}$ defined in this way by $|K|_n$. Since $|K|_{n+1} \supset |K|_n$, we have $\Theta_{n+1} \subset \Theta_n$. So K determines a nest (Θ): $\Theta_0 \supset \Theta_1 \supset \ldots$ of closed E-functors on $\mathfrak{C}$. If M^* is in K, then M^n is in $|K|_n$; so M^n is Θ_n-injective.

We shall call a resolution K of $\mathfrak{C}$ *strong* if, for each $n \geqslant 0$ and each M^* in K, the objects of $|K|_n$ are injective over ker δ_M^n. It follows that ker δ_M^n is a Θ_n-monomorphism. So a strong resolution K is a subclass of the class $K(\Theta)$ of all right acyclic complexes M^* over $\mathfrak{C}$ such that (i) M^n is Θ_n-injective $(n \geqslant 0)$, and (ii) ker δ_M^n is a Θ_n-morphism $(n \geqslant 0)$. Now $K(\Theta)$ is also a strong resolution of $\mathfrak{C}$ since (i) and the inclusions $\Theta_0 \supset \Theta_1 \supset \ldots$ show that each object of $|K(\Theta)|_n$ is Θ_n-injective. So we have proved that any strong resolution K is contained in a strong resolution of the form $K(\Theta)$, where (Θ) is a decreasing sequence of E-functors on $\mathfrak{C}$. These two resolutions endow $\mathfrak{C}$ with the same relative homological algebra, so we can claim that all strong resolutions of categories are determined by E-functors.

We have no example of a resolution that is not strong.

9. The existence of K_k-resolutions of Φ-complexes

We use the notation of the preceding section, and suppose that Θ and Φ are E-functors with sufficient injectives such that $\Theta \supset \Phi$. We shall show that every Φ-complex has a K_k-resolution if $\Theta\cdot\Phi \supset \Phi\cdot\Theta$. First, we see that some condition is necessary for the existence of K_k-resolutions of Φ-complexes.

Suppose that A^* is a Φ-injective representation of A^0 and has a K_k-resolution. By proposition 7·3 it also has a K_θ-resolution. Since K_θ dominates K_k, § 4 shows that for each covariant functor T we have a commutative and exact diagram

$$\begin{array}{ccccc} K_k^k\,TA^1 & \to & K_k^k\,TA^2 & \to & K_k^{k+1}\,TA^0 \\ \downarrow & & \downarrow & & \downarrow \\ K_\theta^k\,TA^1 & \to & K_\theta^k\,TA^2 & \to & K_\theta^{k+1}\,TA^0. \end{array}$$

The cokernels of the left-hand morphisms are $\phi K_k^k\,TA^0$ and $\phi K_\theta^k\,TA^0$. Since derived functors and satellites coincide for left exact functors it follows from (8·1) and (8·2) that we have a commutative diagram

$$\begin{array}{ccc} \phi\theta^k TA^0 & \to & \theta^k\phi TA^0 \\ \| & & \downarrow \\ \phi\theta^k TA^0 & \to & \theta^{k+1} TA^0 \end{array}$$

when T is left exact. In particular when $T = \mathrm{Hom}\,(X,\ \ )$ this gives a relation between Θ and Φ.

To prove the main result of this section we need a lemma which will also be used later:

Lemma 9·1. *If Θ is an E-functor with sufficient injectives, Φ is a closed E-functor, and $\Phi \subset \Theta$, then the following statements are equivalent:*

(i) $\Theta\cdot\Phi \supset \Phi\cdot\Theta$;

(ii) *If P^* is in $\tilde{\Phi}$ and $0 \to P^* \to N^* \to Q^* \to 0$ is a Θ-representation of P^* with N^0 Θ-injective then Q^* is in $\tilde{\Phi}$.*

Proof. First suppose that (i) holds. Consider the diagram

$$0 \to P^* \to N^* \to Q^* \to 0.$$

Write p, q for the images of P^*, Q^* in $\Theta(P^2, P^0)$, $\Theta(Q^2, Q^0)$ and n_i for the image of

$$0 \to P^i \to N^i \to Q^i \to 0 \quad \text{in} \quad \Theta(Q^i, P^i).$$

By lemma 2·1, $p\cdot n_2 = -n_0\cdot q$. By hypothesis $P^* \in \tilde{\Phi}$. So $n_0\cdot q \in \Theta\cdot\Phi(Q^2, P^0)$. Hence there exist x in $\Theta(Y, P^0)$ and y in $\Phi(Q^2, Y)$, for some object Y, such that $n_0\cdot q = x\cdot y$. Since N_0 is Θ-injective the connecting homomorphism

$$\mathrm{Hom}\,(Y, Q^0) \to \Theta(Y, P^0)$$

given by $\alpha \to n_0\alpha$ is an epimorphism. So $x = n_0\alpha$ for some α. Hence $n_0\cdot q = n_0\cdot\alpha y$. Since N_0 is Θ-injective the mapping $\quad \Theta(Q^2, Q^0) \to \Theta^2(Q^2, P^0)$

given by $a \to n_0\cdot a$ is an isomorphism. So $q = \alpha y$. But $y \in \Phi(Q^2, Y)$, and $\alpha \in \mathrm{Hom}\,(Y, Q^0)$. Hence $q \in \Phi(Q^2, Q^0)$, and we have proved that (ii) holds.

Now suppose that (ii) holds. Let $x \in \Phi\cdot\Theta$. Then there exist P^* in $\tilde{\Phi}$ and

$$0 \to P^2 \to N^2 \to Q^2 \to 0$$

in Θ such that their join represents x. Write p and n_2 for their classes in $\Phi(P^2, P^0)$ and $\Theta(Q^2, P^2)$. Then $x = p \cdot n_2$. By lemma 7·1 (i) we can embed the two simple extensions in a Θ-representation $0 \to P^* \to N^* \to Q^* \to 0$ of P^* in which N^0 is a Θ-injective. By lemma 2·1, $p \cdot n_2 = -n_0 \cdot q$, where q is the class of Q^* and n_0 is the class of $0 \to P^0 \to N^0 \to Q^0 \to 0$. But $Q^* \in \Phi$ by hypothesis. So $x = p \cdot n_2$ is in $\Theta \cdot \Phi$. Hence $\Theta \cdot \Phi \supset \Phi \cdot \Theta$.

Proposition 9·1. *If Θ and Φ are E-functors with sufficient Θ and Φ-injectives, $\Theta \supset \Phi$, and $\Phi \cdot \Theta \subset \Theta \cdot \Phi$, then every Φ-complex has a K_k-resolution.*

Proof. By proposition 4·3 it is sufficient to show that each element of $\tilde{\Phi}$ admits enough K_k-resolutions.

Let $A^* \in \tilde{\Phi}$. We use the notation of § 7 and observe that M^{**} is a K_k-resolution of A^* if and only if $\mathbf{R}_i$ is a Θ-injective representation of A^{*i} for $i < k$, and $\mathbf{R}_i$ is a Φ-injective representation of A^{*i} for $i \geqslant k$. Suppose that K_k-resolutions M^{0*}, M^{2*} of A^0, A^2 are given. By lemma 7·1 we can construct successively $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{k-1}$ with the desired properties. Then by applying lemma 9·1 successively to $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{k-1}$ we see that A^{*k} is in $\tilde{\Phi}$. Since there are sufficient Φ-injectives proposition 7·3 shows that there exists a complex $\{M^{1i}\}_{i \geqslant k}$ such that $\{M^{*i}\}_{i \geqslant k}$ is a K_ϕ-resolution of A^{*k}. Thus we have constructed a K_k-resolution M^{**} of A with pre-assigned rows M^{0*}, M^{2*}. Next suppose that M^{**} is a normal resolution of A^*, and M^{0*}, M^{2*} are K_k-resolutions of A^0, A^2. Then by using lemma 7·1 (ii) as in the proof of proposition 7·3 we see that $\mathbf{R}_i$ $(i < k)$ is a Θ-injective representation of A^{*i}. Since A^{*0} $(= A^*)$ is in $\tilde{\Phi}$, lemma 9·1 applied successively to $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{k-1}$ shows that A^{*k} is in $\tilde{\Phi}$. Since A^{*k} has enough K_ϕ-resolutions $\{M^{1i}\}_{i \geqslant k}$ is a K_ϕ-resolution of A^{1k}. So M^{1*} is a K_k-resolution of A^1. Thus we have shown that A^* admits enough K_k-resolutions, and the proposition is proved.

10. Commutation of satellites

Let Θ and Φ be E-functors with sufficient injectives such that $\Theta \supset \Phi$. Since $\Theta \supset \Phi$ there exist τ-transformations

$$\nu_T\colon\ \theta\phi T \to \theta^2 T, \quad \mu_T\colon\ \phi\theta T \to \theta^2 T$$

for any covariant functor T on $\mathfrak{C}$ with values in an abelian category. The principal result of this section is:

Theorem 10·1. *The E-functor Φ is central in Θ if and only if for each covariant functor T defined on $\mathfrak{C}$ there exists a natural isomorphism between $\theta\phi T$ and $\phi\theta T$ such that the diagram*

$$\begin{array}{ccc} \phi\theta T & \cong & \theta\phi T \\ {\scriptstyle \mu_T}\downarrow & & {\scriptstyle \nu_T}\downarrow \\ \theta^2 T & = & \theta^2 T \end{array}$$

commutes.

First we obtain formulae for $\phi\theta\,\mathrm{Hom}\,(X,\ \ )$ and $\theta\phi\,\mathrm{Hom}\,(X,\ \ )$. Write μ_X and ν_X for the natural transformations

$$\mu_X\colon\ \phi\theta\,\mathrm{Hom}\,(X,\ \ ) \to \theta^2\,\mathrm{Hom}\,(X,\ \ ) = \Theta^2(X,\ \ ),$$
$$\nu_X\colon\ \theta\phi\,\mathrm{Hom}\,(X,\ \ ) \to \theta^2\,\mathrm{Hom}\,(X,\ \ ) = \Theta^2(X,\ \ ).$$

Proposition 10·1. (i) *μ_X is an injection with image $\Phi \cdot \Theta(X,\ \ )$.*
(ii) *ν_X is an injection with image $\Theta \cdot \Phi(X,\ \ )$.*

Proof. (i) Let A be an object of $\mathfrak{C}$, and $0 \to A \overset{\alpha}{\to} N \overset{\gamma}{\to} C \to 0$ be a Φ-injective representation of A, and n be its image in $\Phi(C, A)$. Let $0 \to A \to M \to B \to 0$ be a Θ-injective representation of A. Since $\Phi \subset \Theta$ there exist morphisms ξ, η such that

$$\begin{array}{ccccccccc} 0 & \to & A & \to & N & \to & C & \to & 0 \\ & & \| & & \xi\downarrow & & \eta\downarrow & & \\ 0 & \to & A & \to & M & \to & B & \to & 0 \end{array}$$

commutes. So we have a commutative and exact diagram

$$\begin{array}{ccccccc} \Theta(X, N) & \to & \Theta(X, C) & \overset{\partial}{\to} & \Theta^2(X, A) & \to & \Theta^2(X, N) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \Theta(X, B) & \to & \Theta^2(X, A) & \to & 0 \end{array}$$

Since $\theta \operatorname{Hom}(X, \ ) = \Theta(X, \ )$, the exactness of the first row shows that $\phi\theta \operatorname{Hom}(X, A)$ can be identified with $\operatorname{Im} \partial$. Since $\mu_X(A)$ is induced by $\Theta(1_X, \eta)$, the commutativity shows that $\mu_X(A)$ can be identified with the inclusion $\operatorname{Im} \partial \subset \Theta^2(X, A)$. In particular μ_X is an injection. It remains to be shown that $\operatorname{Im} \partial = \Phi \cdot \Theta(X, A)$. Since $\partial x = n \cdot x$, $\operatorname{Im} \partial$ is in $\Phi \cdot \Theta(X, A)$. On the other hand, if $x \cdot y \in \Phi \cdot \Theta(X, A)$ with x in $\Phi(Y, A)$ and y in $\Theta(X, Y)$, then

$$\alpha(x \cdot y) = (\alpha x) \cdot y = 0,$$

since $\alpha x \in \Phi(Y, N)$ and N is Φ-injective. So the exactness of the first row of the diagram shows that $x \cdot y$ is in $\operatorname{Im} \partial$. Hence $\operatorname{Im} \partial = \Phi \cdot \Theta(X, A)$, and (i) is proved.

(ii) Let $0 \to A \to M \to B \to 0$ be a Θ-injective representation of A, and write m for its image in $\Theta(B, A)$. Then we have a commutative diagram

$$\begin{array}{ccc} \Phi(X, B) & \to & \theta\phi \operatorname{Hom}(X, A) \\ \downarrow & & \downarrow \nu_X(A) \\ \Theta(X, B) & \overset{\partial}{\to} & \Theta^2(X, A) \end{array}$$

in which the first column is an inclusion and the rows are connecting homomorphisms. Since M is a Θ-injective the rows are isomorphisms. So ν is an injection. It remains to be shown that the image of $\Phi(X, B)$ in $\Theta^2(X, A)$ is $\Theta \cdot \Phi(X, A)$. We have $\partial x = m \cdot x$. So the image of $\Phi(X, B)$ is contained in $\Theta \cdot \Phi(X, A)$. On the other hand, let $z \in \Theta \cdot \Phi(X, A)$. Then $z = y \cdot x$ where $x \in \Phi(X, Y)$, $y \in \Theta(Y, A)$ for some Y. Since M is Θ-injective the connecting homomorphism

$$\operatorname{Hom}(Y, B) \to \Theta(Y, A)$$

is an epimorphism. So $y = m\alpha$ for some α in $\operatorname{Hom}(Y, B)$. Hence

$$z = m\alpha \cdot x = m \cdot \alpha x = \partial(\alpha x).$$

Now x is in $\Phi(X, Y)$. So αx is in $\Phi(X, B)$. Hence z is in the image of $\Phi(X, B)$ in $\Theta^2(X, A)$. Thus (ii) is proved.

We shall also need for the proof of theorem 10·1 and results in later sections the following proposition which we obtain without assuming the existence of sufficient injectives for Θ or Φ.

PROPOSITION 10·2. *If Θ is a closed E-functor, Φ is an E-functor contained in Θ and $\Theta \cdot \Phi \subset \Phi \cdot \Theta$, then $\Theta^n(\ , N)$ is exact on $\tilde{\Phi}$ for $n \geq 0$ and all Φ-injectives N.*

Proof. Let $A^* \in \tilde{\Phi}$, and a be its image in $\Phi(A^2, A^0)$. Since $\Theta(\ , N)$ is an exact Θ-connected sequence it is sufficient to show that the connecting homomorphisms

$$\Theta^n(A^0, N) \to \Theta^{n+1}(A^2, N)$$

are zero. Let $x \in \Theta^n(A^0, N)$. Then the image of x is $x \cdot a$. Since x is in $\Theta^n(A^0, N)$ there exist $x_1, \ldots, x_n$ in Θ such that $x = x_1 \cdot x_2 \cdot \ldots \cdot x_n$. Now $\Theta \cdot \Phi \subset \Phi \cdot \Theta$. So $x \cdot a = b \cdot y_1 \cdot \ldots \cdot y_n$ with b in $\Phi(Z, N)$ for some Z, and $y_1, \ldots, y_n$ in Θ. Since N is Φ-injective $\Phi(Z, N) = 0$. Hence $x \cdot a = 0$. So the connecting homomorphisms are zero and the proposition is proved.

The last of the preliminaries is to obtain formulae for μ_T, ν_T, $\theta\phi T$, $\phi\theta T$, and $\theta^2 T$. Let $A^0 \in \mathfrak{C}$, and A^* be a Φ-injective representation of A^0. By §7 A^* has a Θ-injective representation

$$0 \to A^* \to M^* \to B^* \to 0.$$

So

$$\begin{aligned} \phi\theta TA^0 &\cong \text{Coker}\,[\theta TA^1 \to \theta TA^2] \\ &\cong \text{Coker}\,[\text{Coker}\,(TM^1 \to TB^1) \to \text{Coker}\,(TM^2 \to TB^2)] \\ &\cong \text{Coker}\,[\text{Coker}\,(TM^1 \to TM^2) \to \text{Coker}\,(TB^1 \to TB^2)]. \end{aligned}$$

Since M^* splits, Coker $[TM^1 \to TM^2]$ vanishes. So we have the formula

$$\phi\theta TA^0 \cong \text{Coker}\,[TB^1 \to TB^2].$$

Since M^0 is a Θ-injective

$$\theta\phi TA^0 \cong \text{Coker}\,[\phi TM^0 \to \phi TB^0] \cong \phi TB^0.$$

Write C^0 for B^0 and let C^* be a Φ-injective representation of C^0. Then

$$\theta\phi TA^0 \cong \text{Coker}\,[TC^1 \to TC^2].$$

Write $D^0 = B^0$ and let D^* be a Θ-injective representation of D^0. By definition

$$\theta^2 TA^0 \cong \text{Coker}\,[TD^1 \to TD^2].$$

Finally we obtain formulae for $\mu_T(A^0)$ and $\nu_T(A^0)$. Write b, c, d for the exact sequences

$$0 \to A^0 \to M^0 \to B^1 \to B^2 \to 0, \quad 0 \to A^0 \to M^0 \to C^1 \to C^2 \to 0,$$
$$0 \to A^0 \to M^0 \to D^1 \to D^2 \to 0$$

obtained by joining $0 \to A^0 \to M^0 \to B^0 \to 0$ to B^*, C^*, D^* respectively. Since D_1 is a Θ-injective, and b, c are Θ-complexes, the identity morphism of A^0 can be covered by complex morphisms β: $b \to d$, and γ: $c \to d$. Then β induces $\mu_T(A^0)$, and γ induces $\nu_T(A^0)$.

PROPOSITION 10·3. *If Θ and Φ are E-functors with sufficient injectives such that $\Theta \supset \Phi$, then the following statements are equivalent.*

(i) $\Theta \cdot \Phi \subset \Phi \cdot \Theta$.

(ii) $\Theta(\ , N)$ *is exact on* $\tilde{\Phi}$ *for all* Φ-*injectives* N.

(iii) *If A is any Φ-injective and $0 \to A \to M \to B \to 0$ is a Θ-injective representation of A, then B is a Φ-injective.*

(iv) *For each covariant functor T defined on $\mathfrak{C}$, there exists a natural transformation of functors $\lambda_T : \theta\phi T \to \phi\theta T$ such that $\mu_T \lambda_T = \nu_T$.*

(v) *Property* (iv) *is valid for all functors* Hom $(X,\ )$.

Proof. It is trivial that (iv) implies (v) and proposition 10·2 shows that (i) implies (ii). Suppose now that (ii) holds. To deduce that (iii) is true it is sufficient to show that Hom (, B) is exact on $\tilde{\Phi}$. Let X^* be in $\tilde{\Phi}$. Then applying Hom (X^*,) to $0 \to A \to M \to B \to 0$ gives a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}(X^0, B) & \to & \Theta(X^2, B) \\ \downarrow & & \downarrow \\ \Theta(X^0, A) & \to & \Theta^2(X^2, A), \end{array}$$

in which the bottom row is the zero morphism by hypothesis, and the second column is an isomorphism since M is Θ-injective. So the first row is the zero morphism. Hence Hom (, B) is exact on $\tilde{\Phi}$, and (iii) is true.

Next suppose that (iii) holds. Since A^1 is a Φ-injective and

$$0 \to A^1 \to M^1 \to B^1 \to 0$$

is a Θ-injective representation of A^1, B^1 is a Φ-injective. By definition C^* is in $\tilde{\Phi}$. So there exists a complex morphism of C^* into B^* covering the identity morphism of C^0 ($= B^0$), and this together with the identity morphism on

$$0 \to A^0 \to M^0 \to B^0 \to 0$$

determines a complex morphism π of c into b. The morphism π determines a morphism of $\theta\phi TA^0$ into $\phi\theta TA^0$. It can be verified that this morphism is determined by A^0, and that it is the value on A^0 of a natural transformation of functors. Define λ_T to be this natural transformation of functors. Since $\beta\pi$ and γ are both complex morphisms of c into d covering the identity morphism of A^0, they are homotopic. So $\mu_T \lambda_T = \nu_T$. Thus (iii) implies (iv).

Finally proposition 10·1 shows that (v) implies (i).

Proposition 10·4. *If Θ and Φ are E-functors with sufficient injectives such that $\Theta \supset \Phi$, then the following statements are equivalent:*

(i) $\Theta \cdot \Phi \supset \Phi \cdot \Theta$.

(ii) *If P^* is in $\tilde{\Phi}$ and $0 \to P^* \to N^* \to Q^* \to 0$ is a Θ-injective representation of P^*, then Q^* is in $\tilde{\Phi}$.*

(iii) *For each covariant functor T defined on $\mathfrak{C}$ there exists a natural transformation of functors $\lambda'_T : \phi\theta T \to \theta\phi T$ such that $\mu_T = \nu_T \lambda'_T$.*

(iv) *Property* (iii) *is valid for all functors* Hom (X,).

Proof. It is trivial that (iii) implies (iv), proposition 10·1 shows that (iv) implies (i) and lemma 9·1 shows that (i) implies (ii). So it remains to be shown that (ii) implies (iii).

Suppose that (ii) holds. Since A^* is in $\tilde{\Phi}$ and

$$0 \to A^* \to M^* \to B^* \to 0$$

is a Θ-injective representation of A^*, B^* is in $\tilde{\Phi}$. By definition C^1 is a Φ-injective. So there exists a complex morphism of B^* into C^* covering the identity on B^0 ($= C^0$), and this together with the identity morphism on $0 \to A^0 \to M^0 \to B^0 \to 0$ gives a morphism π' of b into c. The morphism π' induces a morphism of $\phi\theta TA^0$ into $\theta\phi TA^0$, and it can be verified that this morphism is the value on A^0 of a natural transformation of $\phi\theta T$ into $\theta\phi T$. Define λ'_T to be this natural transformation of functors. Since $\gamma\pi'$ and β are both complex morphisms of b into d covering the identity morphism of A^0 they are homotopic. So $\nu_T \lambda'_T = \mu_T$. Thus (ii) implies (iii), and the proposition is proved.

We can now prove theorem 10·1. From propositions 10·3 and 10·4 it is sufficient to show that λ_T and λ'_T are mutually inverse. Since

$$0 \to A^* \to M^* \to B^* \to 0$$

is a Θ-injective representation of A^* and A^* is a Φ-injective representation of A^0, proposition 10·3 (iii) and proposition 10·4 (ii) show that B^* is a Φ-injective representation of B^0. Hence we may take C^* to be B^*. Then the complex morphisms π, π' inducing λ_T, λ'_T can be chosen to be the identity morphisms. So λ_T and λ'_T are mutually inverse.

N.B. In this and the preceding section we have obtained conditions for $\Theta\cdot\Phi$ to contain or be contained in $\Phi\cdot\Theta$, under assumptions about the existence of injectives for Θ and Φ. These properties can be formulated for projectives by dualizing $\mathfrak{C}$. Since passage to the dual determines an anti-isomorphism of the Yoneda product, it reverses all inclusion between $\Theta\cdot\Phi$ and $\Phi\cdot\Theta$.

11. Exact couples associated with a pair of E-functors

Let Θ, Φ be E-functors on $\mathfrak{C}$ with sufficient injectives such that $\Theta \supset \Phi$. By proposition 7·3 every K_k-resolution has a K_θ-resolution. So from § 5 there exists for each covariant functor T on $\mathfrak{C}$ with values in an abelian category an exact couple functor $(K_k, K_\theta)\, T$. The first exact couple of this functor consists of exact sequences

$$\to \check{K}^{p+1}_k K^{q-1}_\theta\, T \to \check{K}^p_k K^q_\theta\, T \to K^p_k K^q_\theta\, T \to \check{K}^{p+2}_k K^{q-1}_\theta\, T \to$$

defined for each integer q. Denote the morphisms in this exact sequence by α^{pq}_2, β^{pq}_2, and δ^{pq}_2 respectively. By proposition 5·2, β_2 is induced by the natural transformation of a K_k-satellite into its corresponding K_k-derived functor. We now calculate α_2. From (7·10), (8·1), and (8·2) we have

$$\check{K}^p_k K^q_\theta\, T \cong \theta^{p+q} K^0_\theta\, T \qquad (p \leqslant k),$$

and

$$\check{K}^p_k K^q_\theta\, T \cong \theta^k \phi^{p-k} \theta^q K^0_\theta\, T \qquad (p > k).$$

Proposition 11·1. *When $p < k$, α^{pq}_2 is the identity. When $p \geqslant k$, α^{pq}_2 is the natural transformation*

$$\theta^k \phi^{p-k+1} \theta^{q-1} K^0_\theta\, T \to \theta^k \phi^{p-k} \theta^q K^0_\theta\, T$$

induced by the τ-transformation $\qquad \phi\theta^{q-1} K^0_\theta\, T \to \theta^q K^0_\theta\, T.$

Proof. For $p < k$ the result is trivial. So we assume that $p \geqslant k$. Write Y^* for the subcomplex of X^* defined by $Y^i = 0$ $(i < p)$, $Y^i = X^i$ $(i \geqslant p)$, and N^{**} for the subcomplex of M^{**} defined by $N^{ij} = 0$ $(i < p)$, $N^{ij} = M^{ij}$ $(i \geqslant p)$. Then the canonical monomorphism of N^{**} into M^{**} induces a morphism of the exact couples associated with the filtration on the first index. Since N^{**} is a K_θ-resolution of Y^*, and Y^* is a K_ϕ-resolution of $A^p = \operatorname{Ker} \delta^p_X$, this morphism of exact couples gives in particular a commutative diagram

$$\begin{array}{ccc} \check{K}_\phi K^{q-1}_\theta\, TA^p & \to & K^q_\theta\, TA^p \\ \downarrow & & \downarrow \\ \check{K}^{p+1}_k K^{q-1}_\theta\, TA & \to & \check{K}^p_k K^q_\theta\, TA \end{array}$$

in which the columns are isomorphisms. So it is sufficient to prove that the top row of the diagram is a τ-morphism; that is to prove that the proposition is true for $k = 0$, $p = 0$.

Since K_θ dominates K_ϕ there is a τ-transformation of $(K_\phi, K_\theta)\,T$ into $(K_\theta, K_\theta)\,T$. In particular this gives a commutative diagram

$$\begin{array}{ccc} \check{K}_\phi K_\theta^{q-1}\,T & \to & K_\theta^q\,T \\ \downarrow & & \| \\ \check{K}_\theta K_\theta^{q-1}\,T & \to & K_\theta^q\,T. \end{array}$$

The first column is a τ-morphism, and the second row and column are canonical isomorphisms. So the first row is a τ-morphism. Thus the proposition is true for $k = 0$, $p = 0$ and the proof is completed.

The exact couple functor $(K_k, K_\theta)\,T$ determines a filtration on $K_\theta\,T$. Write $F_k^p(K_\theta\,T)$ for the sub-functor of filtration p. By proposition 5·1, $F_k^p(K_\theta^n\,T)$ is the image of $\check{K}_k^p K_\theta^{n-p}\,T$ in $K_\theta^n\,T$ under $\alpha_2^{0p}\alpha_2^{1p}\ldots\alpha_2^{p-1,1}$. So from proposition 11·1 follows:

Proposition 11·2. *For $p > k$*

$$F_k^p(K_\theta^n\,T) = \operatorname{Im}[\theta^k\phi^{p-k}\theta^{n-p}K_\theta^0\,T \to \theta^n K_\theta^0\,T],$$

where the transformation is induced by the τ-transformation

$$\phi^{p-k}\theta^{n-p}K_\theta^0\,T \to \theta^{p-k}\theta^{n-p}K_\theta^0\,T \cong \theta^{n-k}K_\theta^0\,T.$$

For $p \leqslant k$ $$F_k^p(K_\theta^n\,T) = K_\theta^n\,T.$$

Corollary. *If Φ is central in Θ, then $F_k^p(K_\theta^n\,T)$ is the image of $K_{n-p+k}^n K_\theta^0\,T$ in $K_\theta^n\,T$ under the product of the τ-transformation*

$$K_{n-p+k}^n K_\theta^0\,T \to K_\theta^n K_\theta^0\,T$$

and the canonical isomorphism $$K_\theta^n K_\theta^0\,T \cong K_\theta^n\,T.$$

Proof. If $p \leqslant k$ the result is trivial. For § 8 shows that $K_{n-p+k}^n K_\theta^0\,T$ and $K_\theta^n\,T$ are isomorphic. Suppose $p > k$. By theorem 10·1, ϕ and θ commute. So the proposition gives

$$F_k^p(K_\theta^n\,T) \cong \operatorname{Im}[\theta^{n-p+k}\phi^{p-k}K_\theta^0\,T \to \theta^n K_\theta^0\,T],$$

where the transformation is induced by the τ-transformation of $\phi^{p-k}K_\theta^0\,T$ into $\theta^{p-k}K_\theta^0\,T$. Since $K_\theta^0\,T$ is left exact on Θ and Θ contains Φ, we have a commutative diagram

$$\begin{array}{ccc} K_\theta^0\,T & \cong & K_\theta^0\,T \\ \downarrow & & \downarrow \\ K_\phi^0 K_\theta^0\,T & \cong & K_\theta^0 K_\theta^0\,T \end{array}$$

in which the columns are the canonical transformations of the K_ϕ- and K_θ-sequences, and the bottom row is a τ-transformation. Hence $F_k^p(K_\theta^n\,T)$ is the image of the product of the transformation

$$\theta^{n-p+k}\phi^{p-k}K_\phi^0 K_\theta^0\,T \to \theta^n K_\theta^0 K_\theta^0\,T$$

induced by a τ-transformation and the canonical isomorphism between $K_\theta^0 K_\theta^0\,T$ and $\theta^n K_\theta^0\,T$. By 8·5 and 7·10

$$\theta^{n-p+k}\phi^{p-k}K_\phi^0 K_\theta^0\,T \cong K_{n-p+k}^n K_\theta^0\,T \quad \text{and} \quad \theta^n K_\theta^0 K_\theta^0\,T \cong K_\theta^n\,T.$$

So the result follows.

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When T is Hom $(X, \ )$ it can be shown that $F_k^p K_\theta^n TA$ is the class of elements of $\Theta^n(X, A)$ represented by p-fold Θ-extensions

$$0 \to A \overset{\alpha_0}{\to} X_1 \overset{\alpha_1}{\to} X_2 \to \dots \to X_n \overset{\alpha_n}{\to} X \to 0$$

in which α_i $(k \leqslant i < p)$ are Φ-morphisms.

Finally we prove a lemma that will be needed later:

Lemma 11·1. *If* $S = K_\theta^{p-1} T$ $(p > 0)$, *then the transformation*

$$\mu_S\colon\ \phi\theta S \to \theta^2 S$$

(*defined in* § 10) *is a monomorphism.*

Proof. Since $\theta S \cong K_\theta^p T$, $\theta^2 S \cong K_\theta^{p+1} T$, and μ_S is by definition a τ-transformation, proposition 11·1 shows that μ_S can be identified with the transformation $\alpha_2^{0, p+1}$ of $(K_\phi, K_\theta)\ T$. From the exact couple $\alpha_2^{0, p+1}$ is a monomorphism, since its kernel is the image of $K_\phi^{-1} K_\theta^{p+1} T$, which vanishes. So the lemma is proved.

Corollary. *The transformation* λ'_S *defined in proposition* 10·4 *is a monomorphism.*

Proof. By proposition 10·4, λ'_S is a right factor of μ_S. So λ'_S is a monomorphism.

12. Shifting theorems for balanced functors

Let T be a covariant functor of two variables defined on a product of abelian categories $\mathfrak{C} \times \mathfrak{C}'$ with values in an abelian category $\mathfrak{D}$. Let Θ and Θ' be E-functors defined on $\mathfrak{C}$ and $\mathfrak{C}'$ respectively. The functor T will be called (Θ, Θ')-balanced if $T(M, \ )$ is exact on $\tilde{\Theta}'$ and $T(\ , M')$ is exact on $\tilde{\Theta}$ for all Θ-injectives M and Θ'-injectives M'. Dually, T will be called (Θ, Θ')-cobalanced if $T(P, \ )$ is exact on $\tilde{\Theta}'$ and $T(\ , P')$ is exact on $\tilde{\Theta}$ for all Θ-projectives P and Θ'-projectives P'. By dualizing $\mathfrak{C}$, $\mathfrak{C}'$, and $\mathfrak{D}$ we may deduce results for cobalanced functors from results for balanced functors. In most of the applications $\mathfrak{C}$ will be the dual of $\mathfrak{C}'$ and T will be $\mathrm{Hom}_{\mathfrak{C}'}$, or $\mathfrak{C}$ and $\mathfrak{C}'$ will be categories of modules and T will be a tensor product. In this section we shall use the results of § 6 to obtain shifting theorems for $(K_\phi, K_\theta)\ T$. We need two preliminary results on balanced functors.

Proposition 12·1. *If* Θ *and* Θ' *are E-functors defined on* $\mathfrak{C}$ *and* $\mathfrak{C}'$ *with sufficient injectives, and* T *is* (Θ, Θ')-*balanced then the canonical transformations*

$$K_\theta T \to (K_\theta \times K_{\theta'})\ T \leftarrow K_{\theta'} T$$

are isomorphisms.

Proof. By proposition 6·1 it is sufficient to show that

$$K_\theta T \to K_\theta K_{\theta'}^0 T, \quad K_{\theta'} T \to K_{\theta'} K_\theta^0 T$$

are isomorphisms, and $K_\theta^p K_{\theta'}^q T$, $K_{\theta'}^p K_\theta^q T$ vanish for $q > 0$.

Let A be an object of $\mathfrak{C}$ and M^* a Θ-injective resolution of A. Since $T(M, \ )$ is exact on $\tilde{\Theta}'$ when M is a Θ-injective, $T(M^*, \ ) \cong K_{\theta'}^0 T(M^*, \ )$ and $K_{\theta'}^q T(M^*, \ )$ vanishes for $q > 0$. So $K_\theta T \cong K_\theta K_{\theta'}^0 T$ and $K_\theta^p K_{\theta'}^q T$ vanishes for $q > 0$. By symmetry we may interchange Θ and Θ'. So the result is proved.

Proposition 12·2. *Let Φ and Θ be E-functors on $\mathfrak{C}$ with sufficient injectives such that $\Phi \subset \Theta$, and let Φ' and Θ' be E-functors on $\mathfrak{C}'$ with sufficient injectives such that $\Phi' \subset \Theta'$. If $\Theta\cdot\Phi \supset \Phi\cdot\Theta$ and $\Theta'\cdot\Phi' \supset \Phi'\cdot\Theta'$, the functor T is (Θ, Θ')-balanced, and $K^0_\theta T$ is (Φ, Φ')-balanced, then $K^p_\theta T$ is (Φ, Φ')-balanced for all p.*

Proof. Since T is (Θ, Θ')-balanced, $K_\theta T$ and $K_{\theta'} T$ are isomorphic. So the hypotheses and conclusion are symmetric, and it is sufficient to prove that $K^p_\theta T(\ , N')$ is exact on $\tilde{\Phi}$ for all Φ'-injectives N'.

The proof is by induction. By hypothesis $K^0_\theta T(\ , N')$ is exact on $\tilde{\Phi}$. Suppose that $p > 0$, and $K^{p-1}_\theta T(\ , N')$ is exact on $\tilde{\Phi}$. Then $K^p_\theta T(\ , N')$ is left exact on $\tilde{\Phi}$, since $K_\theta T$ is a cohomological Θ-connected sequence and $\Phi \subset \Theta$. So to show that $K^p_\theta T(\ , N')$ is exact on $\tilde{\Phi}$, it is sufficient to show that $\phi K^p_\theta T(\ , N')$ vanishes. Write S for $K^{p-1}_\theta T(\ , N')$. Then θS and $K^p_\theta T(\ , N')$ are isomorphic. So we have to show that $\phi\theta S$ vanishes. Since $\Phi\cdot\Theta \subset \Theta\cdot\Phi$, there is by proposition 10·4 a natural transformation λ'_S: $\phi\theta S \to \theta\phi S$. By the corollary to lemma 11·1, λ'_S is a monomorphism. But $\phi S = 0$, since by the induction hypothesis S is exact on $\tilde{\Phi}$. So $\phi\theta S = 0$. Thus $K^p_\theta T(\ , N')$ is exact on $\tilde{\Phi}$, and the proposition is proved.

The first shifting theorem is:

Theorem 12·1. *Let Θ and Θ' be E-functors on $\mathfrak{C}$ and $\mathfrak{C}'$ with sufficient injectives, and Φ be an E-functor contained in Θ with sufficient injectives. If T is a covariant (Θ, Θ')-balanced functor, then $(K_\phi, K_\theta)\ T \cong (K_\phi * K_{\theta'})\ T$.*

Proof. By proposition 12·1 the canonical transformations

$$K_\theta T \to (K_\theta \times K_{\theta'})\ T \leftarrow K_{\theta'} T$$

are isomorphisms. So from theorem 6·1 $(K_\phi, K_\theta)\ T$ and $(K_\phi * K_{\theta'})\ T$ are isomorphic.

The second shifting theorem is:

Theorem 12·2. *Let Φ and Θ be E-functors on $\mathfrak{C}$ with sufficient injectives such that $\Phi \subset \Theta$, and Φ' and Θ' be E-functors on $\mathfrak{C}'$ with sufficient injectives such that $\Phi' \subset \Theta'$. If*

$$\Phi\cdot\Theta \subset \Theta\cdot\Phi, \quad \Phi'\cdot\Theta' \subset \Theta'\cdot\Phi',$$

the covariant functor T is (Θ, Θ')-balanced, and $K^0_\theta T$ is (Φ, Φ')-balanced, then

$$(K_\phi, K_\theta)\ T \cong (K_\phi * K_{\theta'})\ T \cong (K_{\phi'}, K_{\theta'})\ T \cong (K_{\phi'} * K_\theta)\ T.$$

Proof. Since T is (Θ, Θ')-balanced the preceding theorem shows that the first and third pairs are isomorphic. It remains to show that $(K_\phi * K_{\theta'})\ T$ and $(K_{\phi'}, K_{\theta'})\ T$ are isomorphic. By proposition 12·2, $K_{\theta'} T$ is (Φ, Φ')-balanced. So proposition 12·1 shows that the canonical transformations

$$K_\phi K_{\theta'} T \to (K_\phi \times K_{\phi'})\ K_{\theta'} T \leftarrow K_{\phi'} K_{\theta'} T$$

are isomorphisms. Hence by theorem 6·2 $(K_\phi * K_{\theta'})\ T$ and $(K_{\phi'}, K_{\theta'})\ T$ are isomorphic.

Corollary. *Let Φ and Θ be E-functors on $\mathfrak{C}$ with sufficient projectives and injectives such that $\Phi \subset \Theta$. Write J_ϕ for the class of Φ-projective resolutions of $\mathfrak{C}$. Then*

$$(K_\phi, K_\theta)\ \mathrm{Hom} \cong (K_\phi * J_\theta)\ \mathrm{Hom} \cong (J_\phi, J_\theta)\ \mathrm{Hom} \cong (J_\phi * K_\theta)\ \mathrm{Hom}$$

if and only if Φ is central in Θ.

Proof. Suppose that Φ is central in Θ. In the theorem take $\mathfrak{C}'$ to be the dual of $\mathfrak{C}$, T to be the functor whose value on $A \times B$ is Hom (B, A), and Θ', Φ' to be the duals of Θ, Φ. By the definitions of Θ-injectives and Θ-projectives T is (Θ, Θ')-balanced. Since T is left exact $K^0_\theta T$ and T are isomorphic. So $K^0_\theta T$ is (Φ, Φ')-balanced. The condition $\Phi' \cdot \Theta' \subset \Theta' \cdot \Phi'$ is equivalent to $\Phi \cdot \Theta \supset \Theta \cdot \Phi$, since dualization determines an anti-isomorphism of the Yoneda product. Thus the conditions of the theorem are satisfied and the four exact couples are isomorphic.

Conversely, suppose that the exact couples are isomorphic. Then the isomorphism between (K_ϕ, K_θ) Hom and (J_ϕ, J_θ) Hom, yields in particular a commutative diagram

$$\begin{array}{ccc} \check{K}^1_\phi K^1_\theta \operatorname{Hom} & \to & K^2_\theta \operatorname{Hom} \\ \| & & \| \\ \check{J}^1_\phi J^1_\theta \operatorname{Hom} & \to & J^2_\theta \operatorname{Hom}. \end{array}$$

By proposition 11·1 the first and second rows are τ-transformations. So from proposition 10·1 and its dual, their images are $\Phi \cdot \Theta$ and $\Theta \cdot \Phi$. Hence $\Phi \cdot \Theta = \Theta \cdot \Phi$.

13. Adjoint functors

We shall show how to associate an E-functor with a pair of adjoint functors. First we need a preliminary result on connecting morphisms.

Let T, U be a pair of covariant functors defined on an abelian category $\mathfrak{C}$ with values in an abelian category $\mathfrak{C}'$, and Θ be a closed E-functor on $\mathfrak{C}$. Suppose that for each A^* in $\tilde{\Theta}$ there is a morphism ∂_A of TA^2 into UA^0 such that

$$TA^1 \to TA^2 \to UA^0 \to UA^1$$

is exact, and if α^* is a complex morphism of A^* into B^*, then

$$\partial_B T(\alpha^2) = U(\alpha^0)\, \partial_A.$$

In particular ∂_A depends only on the class a of A^* in $\Theta(A^2, A^0)$. Define a mapping

$$\beta\colon\ \Theta(A^2, A^0) \to \operatorname{Hom}(TA^2, UA^0)$$

by $\beta(a) = \partial_A$. Then β is natural and the usual device of obtaining the sum of two elements from their direct sum by diagonal and codiagonal morphisms shows that β is a homomorphism. Write $\Phi(A^2, A^0)$ for the kernel of β. The naturality of β shows that Φ is an E-functor.

Proposition 13·1. *Φ is a closed E-functor.*

Proof. Let $A^* \in \tilde{\Phi}$. Then for each X in $\mathfrak{C}$ there is a commutative diagram

$$\begin{array}{ccccc} \Theta(X, A^0) & \to & \Theta(X, A^1) & \to & \Theta(X, A^2) \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle\beta}\downarrow & & \\ \operatorname{Hom}(TX, UA^0) & \to & \operatorname{Hom}(TX, UA^1) & & \end{array}$$

in which the first row is exact, since A^* belongs to $\tilde{\Theta}$ and Θ is closed. Since $A^* \in \tilde{\Phi}$, ∂_A is the zero morphism. So $UA^0 \to UA^1$ is a monomorphism, and the bottom row is a monomorphism. Let x be an element of $\Phi(X, A^1)$ with image zero in $\Theta(X, A^2)$. Then x is the image of an element y of $\Theta(X, A^0)$. Since $x \in \Phi(X, A^1)$, $\beta(x) = 0$. So the commutativity of the diagram and the fact that the bottom row is a monomorphism show that $\beta(y) = 0$. Hence $y \in \Phi(X, A^1)$. Thus Φ is right closed (§ 1). Similarly it is left closed, and the proposition is proved.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories, not necessarily abelian, and $F\colon \mathfrak{C} \to \mathfrak{D}$, $G\colon \mathfrak{D} \to \mathfrak{C}$ be covariant functors such that there is a natural bijection

$$\omega\colon \operatorname{Hom}_{\mathfrak{C}}(A, GB) \to \operatorname{Hom}_{\mathfrak{D}}(FA, B),$$

for each A in $\mathfrak{C}$ and B in $\mathfrak{D}$. We say, with Kan (1958), that F is a left adjoint of G and G is a right adjoint of F, and we call F and G a pair of adjoint functors. The naturality of ω means that if α, β, and ξ are in $\operatorname{Hom}_{\mathfrak{C}}(X, A)$, $\operatorname{Hom}_{\mathfrak{D}}(B, Y)$, and $\operatorname{Hom}_{\mathfrak{C}}(A, GB)$ respectively, then

$$\omega(G(\beta)\,\xi\alpha) = \beta\omega(\xi)\,F(\alpha). \tag{13·1}$$

We obtain the definition of a pair of contravariant adjoint functors by dualizing $\mathfrak{C}$ or $\mathfrak{D}$. So two contravariant functors F and G are adjoint functors if there is a natural bijection

$$\omega\colon \operatorname{Hom}_{\mathfrak{C}}(A, GB) \to \operatorname{Hom}_{\mathfrak{D}}(B, FA)$$

for each A in $\mathfrak{C}$ and B in $\mathfrak{D}$, or if there is a natural bijection

$$\omega\colon \operatorname{Hom}_{\mathfrak{C}}(GB, A) \to \operatorname{Hom}_{\mathfrak{D}}(FA, B)$$

for each A in $\mathfrak{C}$ and B in $\mathfrak{D}$. We shall obtain our results for covariant adjoint functors. Corresponding results for contravariant adjoint functors may be deduced by dualizing $\mathfrak{C}$ or $\mathfrak{D}$.

In the remainder of this section we assume that $\mathfrak{C}$ is abelian. Let B be an object of $\mathfrak{D}$, and write $\Phi_B(X, Y)$ for the subclass of $\operatorname{Ext}^1_{\mathfrak{C}}(X, Y)$ represented by simple extensions on which $\operatorname{Hom}_{\mathfrak{C}}(\ , GB)$ is exact. Define Φ by

$$\Phi = \bigcap_{B \in D} \Phi_B.$$

Then we shall prove that Φ is a closed E-functor. By proposition 1·3 it is sufficient to show that Φ_B is a closed E-functor for each B in $\mathfrak{D}$. This follows by applying the dual of proposition 13·1 to the contravariant functors $U = \operatorname{Hom}_{\mathfrak{C}}(\ , GB)$ and $T = \operatorname{Ext}^1_{\mathfrak{C}}(\ , GB)$. So we have proved:

Proposition 13·2. *Φ is a closed E-functor.*

Before obtaining other characterizations of Φ we recall some properties of a pair of adjoint functors obtained by Kan (1958).

Let A be an object of $\mathfrak{C}$. Then we have a bijection

$$\omega\colon \operatorname{Hom}_{\mathfrak{C}}(A, GFA) \to \operatorname{Hom}_{\mathfrak{C}}(FA, FA).$$

Put $\mu_A = \omega^{-1}(1_{FA})$. The naturality of ω shows that μ is a natural transformation of the identity functor into GF: that is, for each A in $\mathfrak{C}$ there is a natural morphism

$$\mu_A\colon A \to GFA.$$

If α belongs to $\operatorname{Hom}_{\mathfrak{C}}(X, A)$, then (13·1) with $\xi = \mu_A$ and $\beta = 1_{FA}$ shows that

$$\omega(\mu_A\alpha) = F(\alpha). \tag{13·2}$$

Lastly, putting $\alpha = \mu_A$, $\xi = 1_{GFA}$, and $\beta = 1_{FA}$ in (13·1) shows that

$$\omega(1_{GFA})\,F(\mu_A) = 1_{FA}. \tag{13·3}$$

Now we obtain other characterizations of Φ.

Proposition 13·3. *If α is a monomorphism the following statements are equivalent:* (i) *α is a Φ-morphism;* (ii) *$F(\alpha)$ has a left inverse;* (iii) *$GF(\alpha)$ has a left inverse.*

Proof. First we shall prove that (i) and (ii) are equivalent. If $\alpha: A \to B$ is a Φ-monomorphism, then by definition

$$\mathrm{Hom}_{\mathfrak{C}}(B, GX) \to \mathrm{Hom}_{\mathfrak{C}}(A, GX)$$

is an epimorphism for all X in $\mathfrak{D}$. By taking $X = FA$ and using the adjointness of F and G we see that

$$\mathrm{Hom}_{\mathfrak{D}}(FB, FA) \to \mathrm{Hom}_{\mathfrak{D}}(FA, FA)$$

is a projection. Hence $F(\alpha)$ has a left inverse. Conversely, if $F(\alpha)$ has a left inverse, then

$$\mathrm{Hom}_{\mathfrak{D}}(FB, X) \to \mathrm{Hom}_{\mathfrak{D}}(FA, X)$$

is a projection for all X in $\mathfrak{D}$. So by the adjointness relation and the definition of Φ, α is a Φ-morphism.

It is clear that (ii) implies (iii). So it remains to be shown that (iii) implies (ii). Since μ is a natural transformation of the identity functor into GF

$$\mu_B \alpha = GF(\alpha)\,\mu_A.$$

So operating with F gives
$$F(\mu_B)\,F(\alpha) = FGF(\alpha)\,F(\mu_A).$$

Since $GF(\alpha)$ has a left inverse, $FGF(\alpha)$ has a left inverse. By (13·3) $F(\mu_A)$ has a left inverse. Hence $F(\alpha)$ has a left inverse. So (iii) implies (ii).

Corollary.
$$\Phi(A, B) = \mathrm{Ker}\,\mathrm{Ext}^1(1_A, \mu_B).$$

Proof. Since GFB is a Φ-injective, $\Phi(A, B) \subset \mathrm{Ker}\,\mathrm{Ext}^1(1_A, \mu_B)$. Let

$$0 \to B \xrightarrow{\beta} X \to A \to 0$$

represent an element of $\mathrm{Ker}\,\mathrm{Ext}^1(1_A, \mu_B)$. Then there exists a morphism γ such that $\gamma\beta = \mu_B$. Hence by (13·3)
$$\omega(1_{GFB})\,F(\gamma)\,F(\beta) = 1_{FB}.$$

So by (ii) β is a Φ-monomorphism. Hence $\mathrm{Ker}\,\mathrm{Ext}^1(1_A, \mu_B) \subset \Phi(A, B)$.

Part (iii) of this proposition shows that Φ is determined by GF. Since the category $\mathfrak{C}$ is abelian it is usually more convenient to use GF rather than F. In particular when GF is exact, (iii) shows that a simple extension A^* is in $\tilde{\Phi}$ if and only if GFA^* splits.

14. The existence of injectives

We use the notation of the preceding section. By definition $\mathrm{Hom}_{\mathfrak{C}}(\ , GB)$ is exact on $\tilde{\Phi}$ for any B in $\mathfrak{D}$. So the objects GB are Φ-injectives. In particular, if A belongs to $\mathfrak{C}$, then GFA is a Φ-injective. From (13·2) it follows that μ_A is a monomorphism if, for each X in $\mathfrak{C}$,

$$F\colon \mathrm{Hom}_{\mathfrak{C}}(X, A) \to \mathrm{Hom}_{\mathfrak{D}}(FX, FA)$$

is an injection. Further, (13·3) shows that $F(\mu_A)$ has a left inverse, and so by proposition 13·3 μ_A is a Φ-monomorphism if it is a monomorphism. Hence we have proved:

PROPOSITION 14·1. *If* F: $\mathrm{Hom}_{\mathfrak{C}}(X, A) \to \mathrm{Hom}_{\mathfrak{D}}(FX, FA)$ *is an injection for each* X *and* A *in* $\mathfrak{C}$, *then each* A *in* $\mathfrak{C}$ *has a* Φ-*injective representation*

$$0 \to A \to GFA \to \mathrm{Coker}\,\mu_A \to 0.$$

When μ_A is not a monomorphism, then we have:

PROPOSITION 14·2. *If* Θ *is an E-functor on* $\mathfrak{C}$ *containing* Φ, *there exist sufficient* Θ-*injectives, and* $\ker \mu_A$ *is a* Θ-*morphism for each* A *in* $\mathfrak{C}$, *then* $\mathfrak{C}$ *has sufficient* Φ-*injectives.*

Proof. Let A belong to $\mathfrak{C}$, and write A' for $\mathrm{Ker}\,\mu_A$. Then there exists a Θ-monomorphism of A' into a Θ-injective A_1, and since A_1 is injective over $\ker \mu_A$ we have a commutative diagram

$$\begin{array}{ccccccc} 0 \to & A' & \to & A & \to & GFA \\ & \downarrow & & \alpha\downarrow & & \| \\ & A_1 & \to & Q & \to & GFA \end{array}$$

in which $Q = A_1 \oplus GFA$, and α is a monomorphism. The commutativity shows that α is a right factor of μ_A. Since $F(\mu_A)$ has a left inverse, $F(\alpha)$ has a left inverse. So by proposition 13·3, α is a Φ-monomorphism. Hence there exist sufficient Φ-injectives.

When an E-functor is defined as an intersection of E-functors we have:

PROPOSITION 14·3. *Let* $\mathfrak{C}$ *be an abelian category with arbitrary direct products. If* $\{\Theta_i\}$ *is a set of E-functors defined on* $\mathfrak{C}$, *and each member has sufficient injectives, then* $\bigcap \Theta_i$ *has sufficient injectives.*

Proof. Let A belong to $\mathfrak{C}$ and μ_i be a Θ_i-monomorphism of A into a Θ_i-injective M_i. Write M for ΠM_i, and let μ be the monomorphism of A into M with components μ_i. Since M_i is a Θ_i-injective, M is a Θ-injective, where Θ is $\bigcap \Theta_i$. Since $\mu_i = \pi_i \mu$, where π_i is the canonical projection of M into M_i, and μ_i is a Θ_i-morphism, axiom (d) for h.f. classes shows that μ is a Θ_i-morphism. Hence μ is a Θ-monomorphism, and the proposition is proved.

15. E-functors defined by a set of objects

Let $\mathfrak{C}$ be an abelian category admitting infinite direct sums and P be an object of $\mathfrak{C}$. Write $\Phi_P(X, Y)$ for the subclass of $\mathrm{Ext}^1(X, Y)$ represented by simple extensions on which $\mathrm{Hom}(P, \;)$ is exact. By applying proposition 13·1 to $T = \mathrm{Hom}(P, \;)$ and $U = \mathrm{Ext}^1(P, \;)$, we see that Φ_P is a closed E-functor. We shall call Φ_P the E-functor with a given projective P. If $\{P_i\}$ is a set of objects, then we call $\bigcap \Phi_{P_i}$ the E-functor with a given set of projectives $\{P_i\}$. Since $\mathrm{Hom}(\Sigma P_i, \;)$ is exact on A^* if and only if each functor $\mathrm{Hom}(P_i, \;)$ is exact on A^*, the E-functor $\bigcap \Phi_{P_i}$ is identical with Φ_Q, where $Q = \Sigma P_i$. So we need only discuss the properties of E-functors defined by a single object.

Let $\mathfrak{D}$ be the category of sets. Then $\mathrm{Hom}_{\mathfrak{C}}(P, \;)$ determines for fixed P a covariant functor from $\mathfrak{C}$ to $\mathfrak{D}$. Denote this functor by F_P. For each B in D define $G_P B$ to be $P^{(B)}$, the direct sum of a set of copies of P indexed by the set B. Then G_P can be regarded as a functor from $\mathfrak{D}$ to $\mathfrak{C}$. By the definition of a direct sum (Grothendieck 1957) there is a natural bijection

$$\mathrm{Hom}_{\mathfrak{C}}(P^{(B)}, A) \to \prod_B \mathrm{Hom}_{\mathfrak{C}}(P, A),$$

where $\prod_B$ denotes the product of a set of copies indexed by B. But $\prod_B$ may be identified with $\mathrm{Hom}_{\mathfrak{D}}(B, \ )$. So we have a natural bijection

$$\mathrm{Hom}_{\mathfrak{C}}(G_P B, A) \to \mathrm{Hom}_{\mathfrak{D}}(B, F_P A).$$

Thus G_P is a left adjoint of F_P and F_P is a right adjoint of G_P, and to apply the theory of §§ 13, 14 it is only necessary to dualize $\mathfrak{C}$ and $\mathfrak{D}$. Let Φ'_P be the E-functor obtained by the construction of § 13, suitably dualized, from F_P and G_P. By definition A^* is in $\tilde{\Phi}'_P$ if and only if $\mathrm{Hom}_{\mathfrak{C}}(G_P B, A^*)$ is exact for all sets B. But $G_P B$ is just a direct sum of copies of P. So A^* is in $\tilde{\Phi}'_P$ if and only if $\mathrm{Hom}_{\mathfrak{C}}(P, A^*)$ is exact. Hence Φ_P and Φ'_P are identical.

From the definition of a generator (Grothendieck 1957, p. 134) the mapping

$$\mathrm{Hom}_{\mathfrak{C}}(A, B) \to \mathrm{Hom}_{\mathfrak{D}}(\mathrm{Hom}_{\mathfrak{C}}(P, A), \mathrm{Hom}_{\mathfrak{C}}(P, B))$$

is an injection if and only if P is a generator of $\mathfrak{C}$. So from proposition 14·1 we deduce that $\mathfrak{C}$ has sufficient Φ_P-projectives when P is a generator. If P is not necessarily a generator, then proposition 14·2 shows that $\mathfrak{C}$ has sufficient Φ_P-projectives if it has sufficient projectives. Hence we have proved:

Proposition 15·1. *There exist sufficient Φ_P-projectives in $\mathfrak{C}$ if*: (i) *P is a generator of $\mathfrak{C}$*; *or*, (ii) *$\mathfrak{C}$ has sufficient projectives.*

As a first application of this construction we show that not all closed E-functors are central.

Lemma 15·1. *If $\mathfrak{C}$ is an abelian category, and A, P are objects of $\mathfrak{C}$ such that* $\mathrm{Ext}^1(P, A) = 0$, $\mathrm{Ext}^2(P, A) \neq 0$, *then Φ_P is not a central E-functor.*

Proof. By proposition 10·2 it is sufficient to find an element of $\tilde{\Phi}_P$ on which $\mathrm{Ext}^1(P, \ )$ is not exact. Let z be a non-zero element of $\mathrm{Ext}^2(P, A)$ and

$$0 \to A \xrightarrow{\alpha} X \to Y \to P \to 0$$

be an extension representing z. Let x and y be the classes of

$$0 \to A \to X \to B \to 0 \ (\tilde{x}), \quad \text{and} \quad 0 \to B \to Y \to P \to 0 \ (\tilde{y}),$$

where $B = \mathrm{Coker}\,\alpha$. We shall show that $\tilde{x}$ is a simple extension with the required properties. Since $\mathrm{Ext}^1(P, A) = 0$, the sequence $\mathrm{Hom}(P, \tilde{x})$ is exact. So $\tilde{x}$ belongs to $\tilde{\Phi}_P$. The sequence

$$\mathrm{Ext}^1(P, X) \to \mathrm{Ext}^1(P, B) \to \mathrm{Ext}^2(P, A)$$

is exact, the image of the element y of $\mathrm{Ext}^1(P, B)$ is xy, and $xy = z \neq 0$. Hence the functor $\mathrm{Ext}^1(P, \ )$ is not exact on $\tilde{x}$. So $\tilde{x}$ has the required properties, and the lemma is proved.

The hypotheses of the lemma can be satisfied by taking $\mathfrak{C}$ to be the category of unitary G-modules, where G is the group of order 2, and the objects P, A both to be the rational integers Z with G acting trivially. Since $\mathrm{Ext}^1(Z, Z)$ vanishes, and $\mathrm{Ext}^2(Z, Z)$ is non-vanishing (Cartan & Eilenberg 1956, p. 251), the lemma shows that Φ_P is not central. Also $\mathfrak{C}$ possesses sufficient projectives and injectives, and by proposition 15·1 Φ_P possesses sufficient projectives. Thus we have constructed an E-functor with sufficient projectives which is not central, on a category with sufficient projectives and injectives.

16. Conditions for Φ to be central in Θ

Let $\mathfrak{C}$ be an abelian category, $\mathfrak{D}$ be a category, and let F, G, and Φ be as in § 13. We shall obtain conditions for Φ to be central in a given E-functor Θ containing it.

Proposition 16·1. *Let Θ be an E-functor containing Φ such that* $\ker \mu_A$ *is a Θ-morphism for all A in $\mathfrak{C}$, and $\mathfrak{C}$ has sufficient Θ-injectives. Then $\Theta \cdot \Phi \subset \Phi \cdot \Theta$ if GFA^* is in $\tilde{\Theta}$ whenever A^* is in $\tilde{\Theta}$.*

Proof. Since Φ has sufficient injectives (by proposition 14·2), proposition 10·3 shows that it is sufficient to prove that, if P is a Φ-injective and

$$0 \to P \to M \to Q \to 0$$

is a Θ-injective representation of P, then Q is a Φ-injective. By the construction of proposition 14·2 there exist Θ-injectives P_1, Q_1 and morphisms ρ, σ of P, Q into P_1, Q_1 such that $\rho \oplus \mu_P$, $\sigma \oplus \mu_Q$ are Φ-morphisms. Since P_1 is a Θ-injective there exists a morphism π of M into $P_1 \oplus Q_1$ making the diagram

$$\begin{array}{ccccccccc} 0 \to & P & \to & M & \to & Q & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & P_1 \oplus GFP & \to & P_1 \oplus Q_1 \oplus GFM & \to & Q_1 \oplus GFQ & \to 0 \end{array}$$

commutative, where the columns are $\rho \oplus \mu_P$, $\pi \oplus \mu_M$, and $\sigma \oplus \mu_Q$. The functor GF transforms $\tilde{\Theta}$ into itself, so the bottom row is in $\tilde{\Theta}$. Since the bottom row is exact $\pi \oplus \mu_M$ is a monomorphism. Furthermore lemma 7·1 (ii) shows that $\pi \oplus \mu_M$ is a Θ-monomorphism. So applying Hom $(X, \ )$ to the above diagram gives an anticommutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\,(X, B) & \to & \Theta(X, A) \\ \downarrow & & \downarrow \\ \Theta(X, Q) & \to & \Theta^2(X, P), \end{array}$$

where A and B are the cokernels of $\rho \oplus \mu_P$ and $\sigma \oplus \mu_Q$. Since P is a Φ-injective, and $\rho \oplus \mu_P$ is a Φ-monomorphism, $\rho \oplus \mu_P$ splits. So the second column of this diagram is the zero morphism. The bottom row is an isomorphism, since M is a Θ-injective. So the anticommutativity shows that the image of Hom (X, B) in $\Theta(X, Q)$ is zero. Since X is arbitrary, this shows that $\sigma \oplus \mu_Q$ splits. Hence Q is a direct factor of a Φ-injective, and the proposition is proved.

Proposition 16·2. *Let Θ be an E-functor with sufficient projectives which contains Φ. Then $\Theta \cdot \Phi \subset \Phi \cdot \Theta$ if GFA^* is in $\tilde{\Theta}$ whenever A^* is in $\tilde{\Theta}$.*

Proof. By the dual of lemma 9·1 it is sufficient to show that, if P^* is in $\tilde{\Phi}$, and

$$0 \to Q^* \to R^* \to P^* \to 0$$

is an exact sequence of Θ-morphisms of simple Θ-extensions in which R^2 is Θ-projective, then Q^* is in $\tilde{\Phi}$. From proposition 13·3 this is equivalent to proving that GFQ^* splits.

Write π^i for the epimorphism of R^i onto P^i. Since R^* is in $\tilde{\Theta}$ and R^2 is a Θ-projective, δ^1_R has a right inverse ρ, say. Also since P^* is in $\tilde{\Phi}$, $F\delta^0_P$ has a left inverse λ, say. The product $\lambda F(\pi^1)\, F(\rho)$ is a morphism of FR^2 into FP^0. First we show that there is a morphism σ in $\mathrm{Hom}_{\mathfrak{D}}\,(FR^2, FR^0)$ such that

$$F(\pi^0)\, \sigma = \lambda F(\pi^1)\, F(\rho).$$

Since $GF\pi^0$ is a Θ-epimorphism, and R^2 is Θ-projective, the mapping

$$\mathrm{Hom}_{\mathfrak{C}}(R^2, GFR^0) \to \mathrm{Hom}_{\mathfrak{C}}(R^2, GFP^0),$$

is an epimorphism. So the adjointness of F and G shows that the mapping

$$\mathrm{Hom}_{\mathfrak{D}}(FR^2, FR^0) \to \mathrm{Hom}_{\mathfrak{D}}(FR^2, FP^0),$$

induced by $F(\pi^0)$ is a projection. Hence σ exists.

From what we have proved the diagram

$$0 \to GFQ^* \to GFR^* \to GFP^* \to 0$$

has the following properties: $GF(\delta_P^0)$ has a left inverse $G(\lambda)$; $GF(\delta_R^1)$ has a right inverse $GF(\rho)$; there exists a morphism $G(\sigma)$ of GFR^2 into GFR^0 such that

$$GF(\pi^0)\,G(\sigma) = G(\lambda)\,GF(\pi^1)\,GF(\rho).$$

So to complete the proof that GFQ^* splits it is sufficient to show:

Lemma 16·1. *If A^*, B^*, C^* are simple extensions in an abelian category, the diagram*

$$0 \to A^* \overset{\mu^*}{\to} B^* \overset{\nu^*}{\to} C^* \to 0$$

is exact and commutative, and there exist morphisms β, γ, and σ such that β is a right inverse of δ_B^1, γ is a left inverse of δ_C^0, and $\nu^0\sigma = \gamma\nu^1\beta$, then A^* splits.

Proof. To show that A^* splits is sufficient to show that δ_B^1 has a right inverse which maps the image of μ^2 into the image of μ^1. Let $\zeta = \beta - \delta_B^0\sigma$. Since $\delta_B^1\delta_B^0 = 0$, and β is a right inverse of δ_B^1, the morphism ζ is a right inverse of δ_B^1. To prove that ζ maps the image of μ^2 into the image of μ^1 it is sufficient to show that $\nu^1\zeta\mu^2 = 0$, for the diagram is exact. We have

$$\begin{aligned}\nu^1\zeta\mu^2 &= \nu^1(\beta - \delta_B^0\sigma)\,\mu^2\\ &= \nu^1\beta\mu^2 - \delta_C^0\nu^0\sigma\mu^2, \quad \text{by commutativity,}\\ &= \nu^1\beta\mu^2 - \delta_C^0\gamma\nu^1\beta\mu^2.\end{aligned}$$

Since γ is a left inverse of δ_C^0 and $\mathfrak{C}$ is abelian, C^* splits. So there exists γ' such that

$$\delta_C^0\gamma + \gamma'\delta_C^1 = 1.$$

Hence
$$\begin{aligned}\nu^1\zeta\mu^2 &= \gamma'\delta_C^1\nu^1\beta\mu^2\\ &= \gamma'\nu^2\delta_B^1\beta\mu^2, \quad \text{by commutativity,}\\ &= \gamma'\nu^2\mu^2, \quad \text{since } \delta_B^1\beta = 1,\\ &= 0.\end{aligned}$$

So the lemma is proved.

Proposition 16·3. *Let Θ be an E-functor containing Φ such that $\mathfrak{C}$ has sufficient Θ-injectives and* $\ker \mu_A$ *is a Θ-morphism for each A in $\mathfrak{C}$. Then Φ is central in Θ if GFA^* is in $\tilde{\Theta}$ whenever A^* is in $\tilde{\Theta}$, and GFM is a Θ-injective whenever M is a Θ-injective.*

Proof. By proposition 16·1 $\Theta\cdot\Phi \subset \Phi\cdot\Theta$. So it remains to be proved that $\Phi\cdot\Theta \subset \Theta\cdot\Phi$. Since $\mathfrak{C}$ has sufficient Θ-injectives, proposition 14·2 shows that it has sufficient Φ-injectives. So by lemma 9·1 it is sufficient to show that if $0 \to P^* \to Q^* \to R^* \to 0$ is a commutative and exact diagram of Θ-morphisms with Q^0 a Θ-injective and P^* in $\tilde{\Phi}$, then R^* is in $\tilde{\Phi}$. Since GF transforms $\tilde{\Theta}$ into itself, the diagram

$$0 \to GFP^* \to GFQ^* \to GFR^* \to 0$$

is a commutative and exact diagram of Θ-morphisms. So by applying Hom $(X, \ )$ we obtain an anticommutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\,(X, GFR^2) & \to & \Theta(X, GFP^2) \\ \downarrow & & \downarrow \\ \Theta(X, GFR^0) & \to & \Theta^2(X, GFP^0). \end{array}$$

Since Q^0 is a Θ-injective and P^* is in $\tilde{\Phi}$, GFQ^0 is a Θ-injective and GFP^* splits. So the bottom row is an isomorphism and the right-hand column is the zero morphism. Therefore the left-hand column is the zero morphism. Hence GFR^* splits. Thus R^* is in $\tilde{\Phi}$, and the proposition is proved.

17. A canonical cohomological E-functor for sheaves

First, we recall some results on morphisms of sheaves of rings from Chevalley (1958/59). Let $\mathscr{R}$ and $\mathscr{S}$ be sheaves of commutative rings over spaces X and Y. Let f be a morphism of $\mathscr{S}$ into $\mathscr{R}$; then f consists of a continuous mapping of Y into X (also denoted by f) together with homomorphisms, which commute with the restriction homomorphisms,

$$f_U\colon\ \mathscr{R}(U) \to \mathscr{S}(f^{-1}(U))$$

defined for each open set U of X, where $\mathscr{R}(U)$ is the ring of sections of $\mathscr{R}$ over U. Write $f_!\mathscr{B}$ for the direct image of an $\mathscr{S}$-module $\mathscr{B}$, and $f^!\mathscr{A}$ for the inverse image of an $\mathscr{R}$-module $\mathscr{A}$. Then the following proposition is known:

Proposition 17·1. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathscr{R}}(\mathscr{A}, f_!\mathscr{B}) \cong \mathrm{Hom}_{\mathscr{S}}(f^!\mathscr{A}, \mathscr{B}).$$

We shall also need a localized form of this proposition. Write $\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, \mathscr{A}')$ for the sheaf of germs of local homomorphisms of an $\mathscr{R}$-module $\mathscr{A}$ into an $\mathscr{R}$-module $\mathscr{A}'$. Then by applying the preceding proposition to the sheaves $\mathscr{A}_U$ and $\mathscr{B}_U$ obtained by extending the restrictions of $\mathscr{A}$ and $\mathscr{B}$ to U and $f^{-1}(U)$ by zero, we deduce

Proposition 17·2. *There is a natural isomorphism*

$$\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, f_!\mathscr{B}) \cong f_!\,\mathscr{H}om_{\mathscr{S}}(f^!\mathscr{A}, \mathscr{B}).$$

Let $\mathfrak{C}$ be the category of $\mathscr{R}$-modules on X. We shall show how to construct an E-functor $\Omega_{\mathscr{R}}$ on $\mathfrak{C}$ which transforms cohomologically; that is f induces a natural transformation of $\Omega_{\mathscr{R}}$ into $\Omega_{\mathscr{S}}$.

Let W be the set of points of X with the discrete topology, and Q be the set of rings $\{\mathscr{R}_x\}$ indexed by X. Then Q is a sheaf of rings over W, and there is a canonical morphism j of Q into $\mathscr{R}$. A Q-module B is a set of $\mathscr{R}_x$-modules B_x indexed by X. From proposition 17·1 there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{R}}(\mathscr{A}, TB) \cong \mathrm{Hom}_Q(U\mathscr{A}, B) = \prod_{x\in X} \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{A}_x, B_x),$$

where $T = j_!$ and $U = j^!$. Let $\Omega_{\mathscr{R}}$ be the E-functor representing simple extensions of $\mathfrak{G}$ on which U splits; that is, $\mathscr{A}^* \in \tilde{\Omega}_{\mathscr{R}}$ if and only if $\mathscr{A}^*_x$ splits for each x in X. The mapping

$$\mathrm{Hom}_{\mathscr{R}}(\mathscr{A},\mathscr{B}) \to \prod_{x\in X} \mathrm{Hom}_{\mathscr{R}_x}(\mathscr{A}_x,\mathscr{B}_x)$$

is an injection. So from proposition 14·1 $\Omega_{\mathscr{R}}$ has sufficient injectives, and

$$\mathscr{A} \to TU\mathscr{A}$$

is an $\Omega_{\mathscr{R}}$-monomorphism of $\mathscr{A}$ into an $\Omega_{\mathscr{R}}$-injective. It is clear that $TU\mathscr{A}$ is the sheaf $C^0(X;\mathscr{A})$ defined in Godement (1958, p. 167). So the resolution obtained by iterating this construction is the 'canonical resolution' defined there.

Now we show that f induces a natural transformation

$$\Omega^p_{\mathscr{R}}(\mathscr{A},\mathscr{B}) \to \Omega^p_{\mathscr{S}}(f^!\mathscr{A},f^!\mathscr{B}).$$

Let $\mathscr{B}^*$ be an $\Omega_{\mathscr{R}}$-injective resolution of $\mathscr{B}$. Since $(f^!\mathscr{B})_y \cong \mathscr{S}_y \otimes_{\mathscr{R}_x} \mathscr{B}_x$, where $x = f(y)$, $f^!\mathscr{B}^*_x$ splits. So $f^!\mathscr{B}^*$ is an acyclic $\Omega_{\mathscr{S}}$-complex over $f^!\mathscr{B}$. Therefore the identity morphism of $f^!\mathscr{B}$ can be covered by a morphism γ of $f^!\mathscr{B}^*$ into an $\Omega_{\mathscr{S}}$-injective resolution of $f^!\mathscr{B}$, and γ is determined up to homotopy. Then γ induces a morphism

$$H^p(\mathrm{Hom}_{\mathscr{S}}(f^!\mathscr{A},f^!\mathscr{B}^*)) \to \Omega^p_{\mathscr{S}}(f^!\mathscr{A},f^!\mathscr{B}),$$

where H^p denotes the operation of forming the pth cohomology group of a complex. Now $f^!$ induces a complex morphism of $\mathrm{Hom}_{\mathscr{R}}(\mathscr{A},\mathscr{B}^*)$ into $\mathrm{Hom}_{\mathscr{S}}(f^!\mathscr{A},f^!\mathscr{B}^*)$, and hence a morphism

$$\Omega^p_{\mathscr{R}}(\mathscr{A},\mathscr{B}) \to H^p(\mathrm{Hom}_{\mathscr{S}}(f^!\mathscr{A},f^!\mathscr{B}^*)).$$

By combining the two morphisms we obtain a morphism of $\Omega^p_{\mathscr{R}}(\mathscr{A},\mathscr{B})$ into $\Omega^p_{\mathscr{S}}(f^!\mathscr{A},f^!\mathscr{B})$ which can be verified to be the value of a natural transformation.

For some sheaves $\mathscr{A}$ the group $\Omega^p_{\mathscr{R}}(\mathscr{A},\mathscr{B})$ is a cohomology group, in fact:

PROPOSITION 17·3. *If $\mathscr{A}$ is pseudo-coherent, $H^p(X; \mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B})) \cong \Omega^p_{\mathscr{R}}(\mathscr{A},\mathscr{B})$.*

Proof. Let $\mathscr{A}$ be pseudo-coherent. Then proposition 4·11 of Grothendieck (1957) states that

$$\mathscr{H}om_Q(U\mathscr{A}, U\mathscr{B}) \cong U\,\mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B}).$$

So together with proposition 17·2 this gives natural isomorphisms

$$\mathscr{H}om_{\mathscr{R}}(\mathscr{A}, TU\mathscr{B}) \cong T\,\mathscr{H}om_Q(U\mathscr{A}, U\mathscr{B}) \cong TU\,\mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B}).$$

Let $\mathscr{B}^*$ be the canonical resolution of $\mathscr{B}$. We have remarked that $\mathscr{B}^*$ is obtained by iterating monomorphisms of the form

$$\mathscr{B} \to TU\mathscr{B},$$

so the isomorphism between $\mathscr{H}om_{\mathscr{R}}(\mathscr{A},TU\mathscr{B})$ and $TU\,\mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B})$ shows that $\mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B}^*)$ is the canonical resolution of $\mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B})$. Since the pth cohomology group of a sheaf $\mathscr{C}$ is the pth cohomology group of the complex obtained by applying the section functor Γ to the canonical resolution of $\mathscr{C}$,

$$H^p(X; \mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B})) \cong H^p(\Gamma\,\mathscr{H}om_{\mathscr{R}}(\mathscr{A},\mathscr{B}^*)).$$

But $\Gamma\,\mathscr{H}om \cong \mathrm{Hom}$. So the second member is $\Omega^p_{\mathscr{R}}(\mathscr{A},\mathscr{B})$, and the proposition is proved.

Finally, we obtain a condition for $\Omega_{\mathscr{R}}$ to be central. The functor U is clearly exact, and T is exact since inductive limits of exact sequences are exact. So TU is exact, and proposition 16·1 with $\Theta = \mathrm{Ext}^1_{\mathscr{R}}$, and $\Phi = \Omega_{\mathscr{R}}$ shows that

$$\Omega_{\mathscr{R}}\cdot\mathrm{Ext}^1_{\mathscr{R}} \supset \mathrm{Ext}^1_{\mathscr{R}}\cdot\Omega_{\mathscr{R}}.$$

We do not know whether the reverse inclusion holds in general. However, suppose that $\mathscr{R}$ is a coherent sheaf of rings. Let $\mathscr{B}$ be an injective sheaf. From Grothendieck (1957) $\mathscr{B}_x$ is an injective $\mathscr{R}_x$-module. So $U\mathscr{B}$ is an injective Q-module. The natural isomorphism

$$\mathrm{Hom}_{\mathscr{R}}(\mathscr{A}, TU\mathscr{B}) \cong \mathrm{Hom}_Q(U\mathscr{A}, U\mathscr{B})$$

and the exactness of U show that $TU\mathscr{B}$ is an injective $\mathscr{R}$-module. We have already seen that TU is exact, so proposition 16·3 shows:

THEOREM 17·1. *$\Omega_{\mathscr{R}}$ is central if $\mathscr{R}$ is coherent.*

18. A canonical homological E-functor for sheaves

Let $\mathscr{R}$ be a sheaf of rings over a space X, and $\mathfrak{C}$ be the category of $\mathscr{R}$-modules. We shall construct an E-functor $\Lambda_{\mathscr{R}}$ which transforms homologically and obtain spectral sequences

$$W^p_{\mathscr{R}}\Lambda^q_{\mathscr{R}}(\mathscr{A},\mathscr{B}) \underset{p}{\Rightarrow} \mathrm{Ext}^n_{\mathscr{R}}(A,B) \quad \text{and} \quad L^p_{\mathscr{R}}\Omega^q_{\mathscr{R}}(\mathscr{A},\mathscr{B}) \underset{p}{\Rightarrow} \mathrm{Ext}^n_{\mathscr{R}}(\mathscr{A},\mathscr{B}),$$

where $L_{\mathscr{R}}$ denotes the class of $\Lambda_{\mathscr{R}}$-projective resolutions and $W_{\mathscr{R}}$ denotes the class of $\Omega_{\mathscr{R}}$-injective resolutions defined in the preceding section.

Let $\mathfrak{D}$ be the category of sheaves of sets over X. To construct $\Lambda_{\mathscr{R}}$ we shall obtain a pair of adjoint functors F, G from $\mathfrak{C}$, $\mathfrak{D}$ to $\mathfrak{D}$, $\mathfrak{C}$. Let F be the functor which associates with an $\mathscr{R}$-module $\mathscr{A}$ the underlying sheaf of sets $F\mathscr{A}$. Denote the set, ring, or module of sections of a sheaf $\mathscr{A}$ of sets, rings, or modules over U by $\mathscr{A}(U)$, and denote the free module generated over a set Z by a ring J by $J\{Z\}$. Let $\mathscr{B}$ be a sheaf of sets. Define $G\mathscr{B}$ to be the $\mathscr{R}$-module associated with the presheaf consisting of the modules $\mathscr{R}(U)\{\mathscr{B}(U)\}$ and the natural restriction homomorphisms. It can be verified that a morphism α of $\mathfrak{D}$ induces a morphism $G(\alpha)$ of $\mathfrak{C}$, and that G is a functor.

PROPOSITION 18·1. *There is a natural bijection.*

$$\omega\colon \mathrm{Hom}_{\mathscr{R}}(G\mathscr{B},\mathscr{A}) \to \mathrm{Hom}_{\mathfrak{D}}(\mathscr{B},F\mathscr{A}).$$

Proof. Let $\rho \in \mathrm{Hom}_{\mathscr{R}}(G\mathscr{B},\mathscr{A})$. Since $G\mathscr{B}$ is determined by the presheaf formed by the modules $\mathscr{R}(U)\{\mathscr{B}(U)\}$ and the natural restriction homomorphisms, ρ is given by a unique family of morphisms

$$\rho_U\colon \mathscr{R}(U)\{\mathscr{B}(U)\} \to \mathscr{A}(U)$$

commuting with the restriction homomorphisms. Since $\mathscr{R}(U)$ has an identity we may identify $\mathscr{B}(U)$ with a subset of $\mathscr{R}(U)\{\mathscr{B}(U)\}$. Let the restriction of ρ_U to this subset be

$$\sigma_U\colon \mathscr{B}(U) \to \mathscr{A}(U).$$

The mappings σ_U commute with the restriction mappings of $\mathscr{B}$, for the ρ_U commute with the restriction homomorphisms of $G\mathscr{B}$. So the family $\{\sigma_U\}$ determines an element σ of $\mathrm{Hom}_{\mathscr{D}}(\mathscr{B},F\mathscr{A})$. Define ω by $\omega(\rho) = \sigma$. Conversely, let $\sigma \in \mathrm{Hom}_{\mathscr{D}}(\mathscr{B},F\mathscr{A})$. Then σ is determined by a family of mappings

$$\sigma_U\colon \mathscr{B}(U) \to \mathscr{A}(U)$$

commuting with the restriction mappings. The σ_U have unique extensions to homomorphisms

$$\rho_U\colon \mathscr{R}(U)\{\mathscr{B}(U)\} \to \mathscr{A}(U),$$

and these homomorphisms commute with the restriction homomorphisms. So the family $\{\rho_U\}$ determines an element ρ of $\mathrm{Hom}_{\mathscr{R}}(G\mathscr{B},\mathscr{A})$. The constructions of ρ from σ and σ from ρ are mutually inverse. So ω is one-one. The naturality of ω can be easily verified.

Let $\Lambda_{\mathscr{R}}$ be the E-functor representing the simple extensions $\mathscr{A}^*$ in $\mathfrak{C}$ for which $F(\delta^1_A)$ has a right inverse. The mapping

$$\mathrm{Hom}_{\mathscr{R}}(\mathscr{B},\mathscr{A}) \to \mathrm{Hom}_{\mathfrak{S}}(F\mathscr{B},F\mathscr{A})$$

is an injection. So from proposition 14·1 $\mathfrak{C}$ has sufficient $\Lambda_{\mathscr{R}}$-projectives, and

$$GF\mathscr{A} \to \mathscr{A}$$

is a $\Lambda_{\mathscr{R}}$-epimorphism of a $\Lambda_{\mathscr{R}}$-projective onto $\mathscr{A}$.

Let f be as in § 17. We show that f induces a natural transformation

$$\Lambda^p_{\mathscr{S}}(\mathscr{A},B) \to \Lambda^p_{\mathscr{R}}(f_!\mathscr{A},f_!\mathscr{B}).$$

Write F' for the functor which maps an $\mathscr{S}$-module to a sheaf of sets on Y. Since $f_!\,F' = Ff_!$, it follows that $Ff_!(\alpha)$ has a right inverse when $F'(\alpha)$ has a right inverse. So $f_!$ is exact on $\tilde{\Lambda}_{\mathscr{S}}$, and maps it into $\tilde{\Lambda}_{\mathscr{R}}$. Let $\mathscr{A}_*$ be a $\Lambda_{\mathscr{S}}$-projective resolution of $\mathscr{A}$. Then $f_!\mathscr{A}_*$ is an acyclic $\Lambda_{\mathscr{R}}$-complex over $f_!\mathscr{A}$. Hence the identity morphism of $f_!\mathscr{A}$ can be covered by a complex morphism γ (determined up to homotopy) of a $\Lambda_{\mathscr{R}}$-projective resolution of $f_!\mathscr{A}$ into $f_!\mathscr{A}_*$. So γ induces a morphism

$$H^p(\mathrm{Hom}_{\mathscr{R}}(f_!\mathscr{A}_*,f_!\mathscr{B})) \to \Lambda^p_{\mathscr{R}}(f_!\mathscr{A},f_!\mathscr{B}).$$

Now $f_!$ induces a complex morphism of $\mathrm{Hom}_{\mathscr{S}}(\mathscr{A}_*,\mathscr{B})$ into $\mathrm{Hom}_{\mathscr{R}}(f_!\mathscr{A}_*,f_!\mathscr{B})$, and hence a morphism

$$\Lambda^p_{\mathscr{S}}(\mathscr{A},\mathscr{B}) \to H^p(\mathrm{Hom}_{\mathscr{R}}(f_!\mathscr{A}_*,f_!\mathscr{B})).$$

By compounding these two morphisms we obtain a morphism of $\Lambda^p_{\mathscr{S}}(\mathscr{A},\mathscr{B})$ into $\Lambda^p_{\mathscr{R}}(f_!\mathscr{A},f_!\mathscr{B})$ which can be verified to be the value of a natural transformation.

Write $K_{\mathscr{R}}$ for the class of injective resolutions of $\mathscr{R}$-modules. We shall obtain the two spectral sequences from:

Proposition 18·2. *The exact couple functors* $(W_{\mathscr{R}},K_{\mathscr{R}})\,\mathrm{Hom}_{\mathscr{R}}$ *and* $(W_{\mathscr{R}}*L_{\mathscr{R}})\,\mathrm{Hom}_{\mathscr{R}}$ *are isomorphic.*

Proof. In the corollary to theorem 6·1 take $L = K_{\mathscr{R}}$, $K' = L_{\mathscr{R}}$, $K = W_{\mathscr{R}}$ and $T = \mathrm{Hom}_{\mathscr{R}}$. To prove the proposition it is sufficient to verify that

$$\mathrm{Hom}_{\mathscr{R}} \to L^0_{\mathscr{R}}\,\mathrm{Hom}_{\mathscr{R}}, \quad \text{and} \quad \mathrm{Hom}_{\mathscr{R}} \to K^0_{\mathscr{R}}\,\mathrm{Hom}_{\mathscr{R}}$$

are isomorphisms, and

$$K^p_{\mathscr{R}}L^q_{\mathscr{R}}\,\mathrm{Hom}_{\mathscr{R}}(\ ,\mathscr{A}) = L^p_{\mathscr{R}}K^q_{\mathscr{R}}\,\mathrm{Hom}_{\mathscr{R}}(\ ,\mathscr{A}) = 0$$

for $q > 0$, and any $\Omega_{\mathscr{R}}$-injective $\mathscr{A}$. Since $\mathrm{Hom}_{\mathscr{R}}$ is left exact the above transformations are isomorphisms. If $\mathscr{M}$ is an injective $\mathscr{R}$-module, then $\mathrm{Hom}_{\mathscr{R}}(\ ,\mathscr{M})$ is exact, and so $L^q_{\mathscr{R}}\,\mathrm{Hom}_{\mathscr{R}}(\ ,\mathscr{M}) = 0$ for $q > 0$. Hence $K^p_{\mathscr{R}}L^q_{\mathscr{R}}\,\mathrm{Hom}_{\mathscr{R}} = 0$, for $q > 0$. To prove that

$L^p_{\mathscr{R}} K^q_{\mathscr{R}} \operatorname{Hom}_{\mathscr{R}}(\;,\mathscr{A})$ vanishes for $\Omega_{\mathscr{R}}$-injectives $\mathscr{A}$, it is sufficient to show that $K^q_{\mathscr{R}} \operatorname{Hom}_{\mathscr{R}}(\mathscr{B},\mathscr{A})$ vanishes when $\mathscr{B}$ is a $\Lambda_{\mathscr{R}}$-projective and $\mathscr{A}$ is an $\Omega_{\mathscr{R}}$-injective. Since $K^q_{\mathscr{R}} \operatorname{Hom}_{\mathscr{R}}$ is $\operatorname{Ext}^q_{\mathscr{R}}$, the proposition follows from:

Lemma 18·1. *If $\mathscr{A}$ is $\Omega_{\mathscr{R}}$-injective and $\mathscr{B}$ is $\Lambda_{\mathscr{R}}$-projective, then* $\operatorname{Ext}^q_{\mathscr{R}}(\mathscr{B},\mathscr{A}) = 0$ *for* $q > 0$.

Proof. Since $\mathscr{A}$ is $\Omega_{\mathscr{R}}$-injective, it is a direct summand of $TU\mathscr{A}$. Similarly $\mathscr{B}$ is a direct summand of $GF\mathscr{B}$. So it is sufficient to show that $\operatorname{Ext}^q_{\mathscr{R}}(G\mathscr{C}, T\mathscr{D})$ vanishes for all sheaves of sets $\mathscr{C}$, and families of $\mathscr{R}_x$-modules $\mathscr{D}$. Let $\mathscr{M}^*$ be an injective resolution of $\mathscr{D}$. Then $T\mathscr{M}^*$ is an injective resolution of $T\mathscr{D}$ (Godement 1958, p. 260). Hence

$$\operatorname{Ext}^q_{\mathscr{R}}(G\mathscr{C}, T\mathscr{D}) \cong H^q(\operatorname{Hom}_{\mathscr{R}}(G\mathscr{C}, T\mathscr{M}^*)) \cong H^q(\Pi \operatorname{Hom}_{\mathscr{R}_x}((G\mathscr{C})_x, \mathscr{M}^*_x)),$$

by the adjointness of T and U. So, since H^q commutes with Π,

$$\operatorname{Ext}^q_R(G\mathscr{C}, T\mathscr{D}) \cong \prod_{x\in X} \operatorname{Ext}^q_{\mathscr{R}_x}((G\mathscr{C})_x, \mathscr{D}_x).$$

Since
$$(G\mathscr{C})_x = \lim_{x\in U} \mathscr{R}(U)\{\mathscr{C}(U)\} = \lim_{x\in U} \mathscr{R}(U)\{\lim_{x\in U} \mathscr{C}(U)\} = \mathscr{R}_x\{\mathscr{C}_x\},$$

it follows that $\operatorname{Ext}^q_{\mathscr{R}_x}((G\mathscr{C})_x, \mathscr{D}_x)$ vanishes. So $\operatorname{Ext}^q_{\mathscr{R}}(G\mathscr{C}, T\mathscr{D})$ vanishes. Thus the lemma and the proposition are proved.

Since the total homology term of $(W_{\mathscr{R}}, K_{\mathscr{R}})$ Hom is $\operatorname{Ext}_{\mathscr{R}}$, it follows from the proposition that the total homology term of $(W_{\mathscr{R}} * L_{\mathscr{R}}) \operatorname{Hom}_{\mathscr{R}}$ is also $\operatorname{Ext}_{\mathscr{R}}$. So the spectral sequence terms of the exact couple $(W_{\mathscr{R}} * L_{\mathscr{R}}) \operatorname{Hom}_{\mathscr{R}}$ give the first of the spectral sequences of this section. Further, $(L_{\mathscr{R}} * W_{\mathscr{R}}) \operatorname{Hom}_{\mathscr{R}}$ and $(W_{\mathscr{R}} * L_{\mathscr{R}}) \operatorname{Hom}_{\mathscr{R}}$ have a common total homology term, and the spectral sequence terms of $(L_{\mathscr{R}} * W_{\mathscr{R}}) \operatorname{Hom}_{\mathscr{R}}$ yield the second of the spectral sequences.

19. The centre of a category

We associate with an abelian category a commutative ring called its centre, and then show that each ideal of the centre determines a pair of adjoint functors.

Let $\mathfrak{C}$ be an abelian category. We call a class $r = \{r_A\}_{A\in\mathfrak{C}}$ of morphisms r_A in $\operatorname{Hom}(A,A)$ such that

$$r_B\xi = \xi r_A \quad \text{for} \quad \xi \text{ in } \operatorname{Hom}(A,B) \tag{19·1}$$

an *endomorphism* of $\mathfrak{C}$. The endomorphisms of $\mathfrak{C}$ form a ring (addition and multiplication being defined by components) of which the zero and identity are the endomorphisms $0 = \{0_A\}_{A\in\mathfrak{C}}$ and $1 = \{1_A\}_{A\in\mathfrak{C}}$, respectively. We denote this ring by $R_{\mathfrak{C}}$, and call it the *centre* of $\mathfrak{C}$. The formula (19·1) implies that the centre is a commutative ring.

Let Θ be an E-functor on $\mathfrak{C}$ and $x \in \Theta^n(A,B)$, where $n \geq 0$. If $r \in R_{\mathfrak{C}}$, the properties of an E-functor show that $r_B x$ and $x r_A$ belong to $\Theta^n(A,B)$.

Lemma 19·1. $r_B x = x r_A$.

Proof. By (19·1) the lemma is true for $n = 0$. Suppose $n = 1$, and let

$$0 \to B \xrightarrow{\beta} X \xrightarrow{\alpha} A \to 0$$

by a representative of x. By (19·1), $\beta r_B = r_X \beta$ and $\alpha r_X = r_A \alpha$. So the diagram

$$\begin{array}{ccc} \operatorname{Hom}(A,A) & \rightarrow & \Theta(A,B) \\ {\scriptstyle r_A}\downarrow & & {\scriptstyle r_B}\downarrow \\ \operatorname{Hom}(A,A) & \rightarrow & \Theta(A,B) \end{array}$$

commutes, and equating the images of 1_A shows that $r_B x = x r_A$. Now suppose that $n > 1$. Put $x = y\cdot z$ with z in $\Theta(A,C)$ and y in $\Theta^{n-1}(C,B)$. Then

$$x r_A = y\cdot z r_A = y\cdot r_A z = y r_A \cdot z \quad \text{and} \quad r_B x = r_B y\cdot z.$$

Since y is in $\Theta^{n-1}(C,B)$ the lemma follows by induction on n.

For each pair of objects A, B we can convert the abelian group $\Theta^n(A,B)$ $(n \geqslant 0)$ into a left $R_{\mathfrak{C}}$-module by defining rx to be $r_B x$ ($r \in R_{\mathfrak{C}}$ and $x \in \Theta^n(A,B)$). Similarly, setting $xr = xr_A$, we endow $\Theta^n(A,B)$ with the structure of a right $R_{\mathfrak{C}}$-module. By lemma 19·1

$$rx = r_B x = x r_A = xr. \tag{19·2}$$

It follows that we may refer without ambiguity to the $R_{\mathfrak{C}}$-module structure of $\Theta^n(A,B)$.

Finally, we show that this module structure is natural in the sense that it commutes with the operations of addition and (Yoneda) multiplication in the ringoid of Θ. First, let $r \in R_{\mathfrak{C}}$ and $x, y \in \Theta^n(A,B)$. Then $x+y$ is defined, and

$$r(x+y) = r_B(x+y) = r_B x + r_B y = rx + ry.$$

Next, let $x \in \Theta^n(A,B)$ and $y \in \Theta^m(B,C)$; then $y\cdot x \in \Theta^{m+n}(A,C)$. Using (19·2) one verifies easily that $r(y\cdot x) = (ry)\cdot x = y\cdot(rx)$ whenever $r \in R_{\mathfrak{C}}$ (their common value will usually be denoted by $ry\cdot x$). So it follows from § 1 that the homomorphisms in the connected sequences (over $\tilde{\Theta}$) of $\{\Theta^n\}_{n\geqslant 0}$ are homomorphisms relative to the $R_{\mathfrak{C}}$-module structure. We state this fact as:

Proposition 19·1. *$\{\Theta^n\}_{n\geqslant 0}$ is a Θ-connected sequence of functors with values in the category of $R_{\mathfrak{C}}$-modules.*

Let I be an ideal of a subring R of $R_{\mathfrak{C}}$ which has a *set* of generators. If $\mathfrak{C}$ admits arbitrary direct sums and products (or if I is finitely generated), we can associate with each object A

$$\text{an epimorphism} \quad \kappa^I(A)\colon \quad A \rightarrow A^I = \inf_{r\in I} \operatorname{Coker} r_A$$

and

$$\text{a monomorphism} \quad \kappa_I(A)\colon \quad \inf_{r\in I} \operatorname{Ker} r_A = A_I \rightarrow A.$$

It is easy to verify that A^I, A_I are the values of covariant additive functors F, G: $\mathfrak{C} \rightarrow \mathfrak{C}$, and that F is right exact and G left exact.

Proposition 19·2. *There is a natural isomorphism*

$$\omega\colon \operatorname{Hom}(A^I, B) \rightarrow \operatorname{Hom}(A, B_I),$$

defined for all pairs of objects A and B in $\mathfrak{C}$.

Proof. Let $\alpha \in \operatorname{Hom}(A^I, B)$. Then $r\alpha\kappa^I(A) = \alpha\kappa^I(A)\, r$ for $r \in R$. The definition of A^I implies that $\kappa^I(A)\, r = 0$ for all r in I. Hence $r\alpha\kappa^I(A) = 0$ for all r in I. Then it follows from the definition of $\kappa_I(B)$ that there is a unique morphism $\omega(\alpha)$ of $\operatorname{Hom}(A, B_I)$ for which

$\kappa_I(B)\,\omega(\alpha) = \alpha\kappa^I(A)$. This defines ω, and by a similar construction one obtains a map ω': $\mathrm{Hom}\,(A, B_I) \to \mathrm{Hom}\,(A^I, B)$ such that $\kappa_I(B)\,\beta = \omega'(\beta)\,\kappa^I(A)$ for all β in $\mathrm{Hom}\,(A, B_I)$. The proof that ω is a natural homomorphism with ω' as a two-sided inverse is easy, and we omit it.

An object A such that $\kappa^I(A)$ and $\kappa_I(A)$ are isomorphisms will be called *I-trivial*. Clearly, for all objects A, both A_I and A^I are I-trivial, so we may identify the class of I-trivial objects of $\mathfrak{C}$ with the classes $\{A^I\colon A \in \mathfrak{C}\}$ and $\{A_I\colon A \in \mathfrak{C}\}$. Let Ψ_I be the E-functor with the I-trivial objects as injectives, and Ψ^I be the E-functor with the I-trivial objects as projectives. Then Ψ_I is the E-functor with the GA, for all A in $\mathfrak{C}$, as injectives. By proposition 19·2 F and G are adjoint functors. So Ψ_I is the E-functor obtained by the construction of § 13. Similarly, since Ψ^I is the E-functor with the FA as projectives it may be obtained by the dual construction.

Put $IB = \sup_{r\in I} \mathrm{Im}\, r_B$. Then IB is the kernel of $\kappa^I(B)$. Hence the sequence

$$\mathrm{Ext}^1\,(A, IB) \to \mathrm{Ext}^1\,(A, B) \to \mathrm{Ext}^1\,(A, B^I)$$

is exact. So from the corollary to proposition 13·3

$$\Psi^I(A, B) = \mathrm{Im}\,[\mathrm{Ext}^1\,(A, IB) \to \mathrm{Ext}^1\,(A, B)]. \tag{19·3}$$

Similarly
$$\Psi_I(A, B) = \mathrm{Im}\,[\mathrm{Ext}^1\,(AI, B) \to \mathrm{Ext}^1\,(A, B)], \tag{19·4}$$

where $AI = \sup_{r\in I} \mathrm{Coim}\, r_B$.

Let S be a ring with centre Z and $\mathfrak{C}$ be the category of left S-modules. Every element z of Z determines a 'left translation' z_A in $\mathrm{Hom}_S\,(A, A)$, and $\{z_A\}$ belongs to $R_{\mathfrak{C}}$. It can be easily verified that the mapping $z \to \{z_A\}$ is a monomorphism of rings. That it is an isomorphism can be proved by observing that an element of $R_{\mathfrak{C}}$ is determined by its effect on the free S-modules (for any module can be expressed as a factor of a free module), and this is determined by its effect on S. So $R_{\mathfrak{C}}$ may be identified with Z and R with a subring of Z. It can be verified that

$$A^I = R/I \otimes_R A, \quad IA = \mathrm{Im}\,[I \otimes_R A \to A],$$
$$A_I = \mathrm{Hom}_R\,(R/I, A), \quad AI = \mathrm{Coim}\,[A \to \mathrm{Hom}_R\,(I, A)].$$

20. Hereditary categories

Let R be a subring of the centre of an abelian category $\mathfrak{C}$ and I be an ideal of R. We assume that R is a set. Put

$$I\,\mathrm{Ext}^1 = \mathrm{Im}\,[I \otimes \mathrm{Ext}^1 \to \mathrm{Ext}^1], \tag{20·1}$$

where the homomorphism is given by $r \otimes a \to ra$. Then $I\,\mathrm{Ext}^1$ is an E-functor. We shall relate $I\,\mathrm{Ext}^1$ with the E-functors Ψ^I and Ψ_I defined in § 19. If $\mathrm{Ext}^2_{\mathfrak{C}}$ vanishes, then we call $\mathfrak{C}$ *hereditary*. The main theorem of this section is:

THEOREM 20·1. *If I is a direct summand of R, or I is finitely generated and projective and $\mathfrak{C}$ is hereditary, then* $\Psi^I = I\,\mathrm{Ext}^1 = \Psi_I$.

First we construct a theory of tensor products of R-modules and objects of $\mathfrak{C}$. We call an R-module M *pseudo-noetherian* if there exists a homomorphism with cokernel M of one finitely generated free module into another; that is, M has a finite set of generators whose relations are finitely generated. The pseudo-noetherian R-modules together with their R-homomorphisms form an additive category. When R is noetherian this category is also abelian, but we shall not need this restriction on R.

PROPOSITION 20·1. *Let $A \in \mathfrak{C}$. There exists a unique additive covariant functor T_A from the category of pseudo-noetherian R-modules to $\mathfrak{C}$ such that:*
(i) $T_A R = A$; (ii) *if* $\rho \in \mathrm{Hom}_R(R, R)$, *then* $T_A(\rho) = \rho(1)_A$; (iii) T_A *is right exact.*

Proof. Define $T_A R$ and $T_A(\rho)$ for ρ in $\mathrm{Hom}_R(R, R)$ by (i) and (ii). Then T_A is an additive covariant functor on the category consisting of R and $\mathrm{Hom}_R(R, R)$. For

$$(\rho+\sigma)(1)_A = (\rho(1)+\sigma(1))_A = \rho(1)_A+\sigma(1)_A,$$

and
$$(\rho\sigma)(1)_A = \rho(\sigma(1))_A = (\rho(1)\,\sigma(1))_A = \rho(1)_A\,\sigma(1)_A.$$

Next T_A extends uniquely to a covariant additive functor on the category of finitely generated free R-modules. Let M be a pseudo-noetherian R-module. Then it is the cokernel of a homomorphism δ of finitely generated free R-modules. By the standard theorems on lifting morphisms to free complexes $\mathrm{Coker}\, T_A(\delta)$ is independent of the choice of δ. Define $T_A M$ to be $\mathrm{Coker}\, T_A(\delta)$. If α is a morphism of pseudo-noetherian R-modules, then define $T_A(\alpha)$ to be the morphism induced by lifting α. The same theorems show that $T_A(\alpha)$ is determined up to isomorphism, and T_A is an additive covariant functor.

To see that (iii) holds we use the theory of Cartan & Eilenberg (1956, chap. 5, § 5). The definition of $L_0 T_A$ shows that $L_0 T_A M \cong \mathrm{Coker}\, T_A(\delta)$. Hence $L_0 T_A \cong T_A$. So T_A is right exact.

To prove uniqueness, let U_A be a covariant additive functor which satisfies (i), (ii) and (iii). Since it satisfies (i) and (ii) it coincides with T_A on the category of finitely generated free R-modules. Hence $L_0 U_A$ and $L_0 T_A$ are isomorphic. But U_A and T_A are right exact. So $U_A \cong T_A$.

PROPOSITION 20·2. *Let A, B belong to $\mathfrak{C}$. There exists a natural transformation of functors χ (depending on A, B) given by homomorphisms*

$$\chi_M\colon\ M \otimes \mathrm{Ext}^r(A, B) \to \mathrm{Ext}^r(A, T_B M)$$

for each pseudo-noetherian R-module M. If M is finitely generated and projective, then χ_M is an isomorphism.

Proof. It is sufficient to define χ_R. For then χ can be defined on the finitely generated free R-modules by addition of components, and on the pseudo-noetherian R-modules by expressing them as factors of finitely generated free R-modules. Define χ_R by $\chi_R(r \otimes a) = ra$. Let $\rho \in \mathrm{Hom}_R(R, R)$. Then

$$\chi_R(\rho \otimes 1)(r \otimes a) = \chi_R(\rho(r) \otimes a) = \rho(r)\,a = \rho(1)_B\, ra = T_B(\rho)\,\chi_R(r \otimes a).$$

So χ_R is a natural transformation. It is clearly a homomorphism. Thus the existence of χ is proved.

Since R has an identity χ_R is an isomorphism. So χ is an isomorphism for any finitely generated free module. Let M be finitely generated and projective. It is direct summand of a finitely generated free R-module L, and so pseudo-noetherian. Hence χ_M is defined. Since χ_L is an ismorphism, and M is a direct summand of L, χ_M is an isomorphism.

Let I be an ideal of R with a finite set of generators r^i $(i = 1, \ldots, N)$. Then $IA = \sup \operatorname{Im} r^i_A$. We prove:

PROPOSITION 20·3. *If I is pseudo-noetherian, then $IA = \operatorname{Im} T_A(j)$, where j is the inclusion of I in R.*

Proof. Let M be the free R-module with a basis m^i $(i = 1, \ldots, N)$. Define a homomorphism p of M onto I by $p(m^i) = r^i$, and put $q = jp$. Since T_A is right exact, $T_A(p)$ is an epimorphism. So $\operatorname{Im} T_A(q) = \operatorname{Im} T_A(j)$. But $\operatorname{Im} T_A(q)$ is $\sup \operatorname{Im} T_A(q^i)$, where q^i is the restriction of q to Rm^i. By proposition 20·1 (ii), $\operatorname{Im} T_A(q^i) = \operatorname{Im} r^i_A$. So the proposition is proved.

We have now an adequate theory of tensor products for the proof of theorem 20·1. From proposition 20·2 we have for each pseudo-noetherian ideal I a commutative diagram

$$\left.\begin{array}{ccc} I \otimes \operatorname{Ext}^1(A,B) & \rightarrow & R \otimes \operatorname{Ext}^1(A,B) \\ \chi_I \downarrow & & \chi_R \downarrow \\ \operatorname{Ext}^1(A, T_B I) & \rightarrow & \operatorname{Ext}^1(A,B) \end{array}\right\} \tag{20·2}$$

in which χ_R is an isomorphism. The bottom row has a factorization

$$\operatorname{Ext}^1(A, T_B I) \xrightarrow{\alpha} \operatorname{Ext}^1(A, IB) \rightarrow \operatorname{Ext}^1(A,B).$$

From (20·1) the image of $I \otimes \operatorname{Ext}^1(A,B)$ in $\operatorname{Ext}^1(A,B)$ is $I\operatorname{Ext}^1(A,B)$, and from (19·3) the image of $\operatorname{Ext}^1(A, IB)$ in $\operatorname{Ext}^1(A,B)$ is $\Psi'^I(A,B)$. So $\Psi'^I(A,B)$ and $I\operatorname{Ext}^1(A,B)$ coincide if $\alpha\chi_I$ is an epimorphism. Proposition 20·2 gives a criterion for χ_I to be an isomorphism. It remains for us to obtain a criterion for α to be an epimorphism. The homomorphism α is induced by the morphism coim $T_B(j)$, where j is the inclusion of I in R. Write $\operatorname{Tor}^R_1(R/I, B)$ for Ker $T_B(j)$. By applying Hom $(A,\ \ )$ to the exact sequence

$$0 \rightarrow \operatorname{Tor}^R_1(R/I, B) \rightarrow T_B I \rightarrow IB \rightarrow 0,$$

we obtain an exact sequence

$$\operatorname{Ext}^1(A, T_B I) \xrightarrow{\alpha} \operatorname{Ext}^1(A, IB) \rightarrow \operatorname{Ext}^2(A, \operatorname{Tor}^R_1(R/I, B)).$$

So α is an epimorphism if $\operatorname{Ext}^2(A, \operatorname{Tor}^R_1(R/I,B)) = 0$. Thus we have proved:

PROPOSITION 20·4. *$\Psi'^I(A,B) = I\operatorname{Ext}^1(A,B)$ if I is a finitely generated projective ideal and* $\operatorname{Ext}^2(A, \operatorname{Tor}^R_1(R/I,B)) = 0$.

The corresponding result for Ψ'_I follows by replacing $\mathfrak{C}$ by its dual $\mathfrak{D}$, and observing that $R_{\mathfrak{C}} = R_{\mathfrak{D}}$ and $I\operatorname{Ext}^1_{\mathfrak{C}} = I\operatorname{Ext}^1_{\mathfrak{D}}$. So with $\operatorname{Ext}^1_R(R/I, B)$ for the dual of $\operatorname{Tor}^R_1(R/I,B)$, we have:

PROPOSITION 20·5. *$\Psi'_I(A,B) = I\operatorname{Ext}^1(A,B)$ if I is a finitely generated projective ideal and* $\operatorname{Ext}^2(\operatorname{Ext}^1_R(R/I,B), A) = 0$.

We now deduce theorem 20·1 from propositions 20·4 and 20·5. Suppose $\mathfrak{C}$ is hereditary and I is a finitely generated projective ideal. The conditions of these propositions hold for all A and B in $\mathfrak{C}$. So Ψ'^I, Ψ'_I and $I\operatorname{Ext}^1$ are identical. Now suppose I is a direct summand of R, and let j be the inclusion monomorphism of I into R. Then $T_B(j)$ is a monomorphism.

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So $\mathrm{Tor}_1^R(R/I, B)$, which was defined to be $\mathrm{Ker}\, T_B(j)$, vanishes. Similarly $\mathrm{Ext}_R^1(R/I, B)$ vanishes. Hence the two propositions show that Ψ^I, Ψ_I and $I\,\mathrm{Ext}^1$ are identical.

Let $\mathfrak{C}$ be the category of modules over a ring S. In the preceding section it has been shown that $R_{\mathfrak{C}}$ is the centre of S. By the uniqueness statement of proposition 20·1 the functors T_A and $\otimes_R A$ can be identified. So $\mathrm{Tor}_1^R(R/I, B)$ has the usual meaning, for it was defined to be the kernel of $T_B(j)$, where j is the inclusion monomorphism of I in R. Similarly $\mathrm{Ext}_R^1(R/I, B)$ has the usual meaning. The diagram 20·2 is valid without restricting I to be pseudo-noetherian, since tensor products are defined for all R-modules, but we cannot relax the condition that I is finitely generated in propositions 20·4 and 20·5.

Let S be a Dedekind domain—that is, a hereditary integral domain. Choose $R = S$. Each ideal of R is projective and finitely generated (Eilenberg & Cartan 1956, chap. VII, §§ 3, 5). Let $\mathfrak{S}$ be a non-empty subset of non-zero ideals of R. By theorem 20·1 the E-functors Ψ^I, Ψ_I, and $I\,\mathrm{Ext}^1$ have a common value. Write Ψ for the common value of

$$\bigcap_{I\in\mathfrak{S}} \Psi^I, \quad \bigcap_{I\in\mathfrak{S}} \Psi_I \quad \text{and} \quad \bigcap_{I\in\mathfrak{S}} I\,\mathrm{Ext}^1.$$

Let $\mathfrak{S}^*$ be the set of ideals of R with the property: J belongs to $\mathfrak{S}^*$ if and only if it contains an ideal of $\mathfrak{S}$. We shall prove that:

(1) Ψ is the E-functor with all R/J ($J \in \mathfrak{S}^*$) as projectives;

(2) Ψ is the E-functor with all R/J ($J \in \mathfrak{S}^*$) as injectives;

(3) P is Ψ-projective if and only if P is a direct sum of ideals of R and modules isomorphic to R/J, where $J \in \mathfrak{S}^*$;

(4) Q is Ψ-injective if and only if Q is a direct factor of a direct product of injectives and modules isomorphic to R/J, where $J \in \mathfrak{S}^*$;

(5) $\Psi^2 = 0$.

Proof. Let Σ, Π be the E-functors with all R/J ($J \in \mathfrak{S}^*$) as projectives, injectives respectively. Then (1) and (2) are equivalent to $\Psi = \Sigma$ and $\Psi = \Pi$. Write $C(\mathfrak{S})$ for the class of all modules A which are I-trivial for some I in $\mathfrak{S}$. Then, from the definitions, Ψ is the E-functor with the modules of $C(\mathfrak{S})$ as projectives, or as injectives. If $J \in \mathfrak{S}^*$ then $R/J \in C(\mathfrak{S})$. Hence Σ-projectives are Ψ-projective, and Π-injectives are Ψ-injective. So we obtain inclusions

$$\Psi \subset \Sigma, \quad \Psi \subset \Pi.$$

To obtain the reverse inclusions we use the theory of modules over Dedekind domains. Let $A \in C(\mathfrak{S})$ and I be an ideal of $\mathfrak{S}$ such that A is I-trivial. Then A may be regarded as a module over R/I. By hypothesis $I \neq 0$. Hence (Zariski & Samuel 1958) R/I is a principal ideal domain, and has only a finite set J_i/I ($i = 1, \ldots, k$) of distinct ideals, where the J_i are ideals of R which contain I and, hence, belong to $\mathfrak{S}^*$. An R/I-module is a direct sum of copies of the $(R/I)/(J_i/I) \cong R/J_i$ (Kaplansky 1954, theorem 6). So A may be expressed as a direct sum

$$A = A_1 \oplus \ldots \oplus A_k,$$

where A_i is a direct sum of a set $|A_i|$ of copies of R/J_i. Hence A is Σ-projective. Since A is an arbitrary member of $C(\mathfrak{S})$ it follows that Ψ-projectives are Σ-projectives, and so $\Psi \supset \Sigma$. Now let $\bar{A}_i$ be a direct product of a set $|A_i|$ of copies of R/J_i. Then $\bar{A}_i$ is again a direct sum of copies of R/J_i (by the preceding structure theorem of Kaplansky), so A_i is one of its direct summands. Hence A is a direct summand of the Π-injective module $\prod_{i=1}^{k} \bar{A}_i$, from which we deduce that $\Psi \supset \Pi$.

This completes the proof of (1) and (2). We use (1) to prove (3), (5). By (1), proposition 15·1 (ii), and the projective forms of propositions 14·2, 14·3, each module A has a Ψ-projective representation

$$0 \to B \to T \oplus U \to A \to 0$$

in which T is the direct sum of copies of R/J ($J \in \mathfrak{S}^*$) and U is projective. A projective module over an hereditary ring is a direct sum of ideals (Cartan & Eilenberg 1956, chap. I). Hence the Ψ-projective $T \oplus U$ has the structure asserted by (3). Furthermore we may use theorem 4 in (Kaplansky 1952) to show that each submodule of $T \oplus U$ is a direct sum of ideals of R and of copies of R/J ($J \in \mathfrak{S}^*$). So each submodule of $T \oplus U$ is Ψ-projective. This fact implies (5). Finally, let A be Ψ-projective. Then A is isomorphic to a submodule of $T \oplus U$ so it has the structure asserted in (3).

(4) follows from (2) just as (3) from (1).

Theorem 20·1, for the category of modules over a Dedekind domain, is essentially due to Nunke (1959). His discussion uses the fact that ideals of a Dedekind domain are inversible in the quotient field.

Let R be the ring of integers and $\mathfrak{S}$, hence $\mathfrak{S}^*$, be the set of its non-zero ideals. Then $\Psi = \bigcap_{n \geq 1} n \operatorname{Ext}^1$. (1) implies that $\tilde{\Psi}(A, B)$ is the class of 'pure' exact sequences

$$0 \to B \to C \to A \to 0$$

in Prufer's sense that

$$nC \cap \operatorname{Im}(B \to C) = \operatorname{Im}(nB \to C)$$

for all integers n. The equivalence of these two characterizations of Ψ is proved differently by Fuchs (1958, § 63). The form (3) of 'pure' projectives is well-known. The description (4) of 'pure' injectives, and a proof that there are sufficient, is due to Łoś (1957).

In the next section we give a further characterization of $\tilde{\Psi}$ as a class of generalized pure extensions.

21. Pure submodules and E-functors

The theory of linear equations over modules (Kertesz 1957) forms the basis for a number of generalizations of Prufer's notion of a pure subgroup of an abelian group. We shall express these generalizations in the language of E-functors.

Let R be a ring and F a fixed free (left unitary R-) module. We write μ_G for the inclusion map of a submodule G of F into F.

If G is a submodule of F, a *G-system of linear equations over a module A* is an element ϕ of $\operatorname{Hom}(G, A)$. The system ϕ is said to be solvable in A if ϕ belongs to the image of $\operatorname{Hom}(\mu_G, 1_A)$. If Γ is a family of submodules G of F, we say that a simple extension A^* is Γ-pure, or that $\delta_A^0 A^0$ is a Γ-pure submodule of A^1, if

$$\operatorname{Im}\operatorname{Hom}(\mu_G, \delta_A^0) = \operatorname{Im}\operatorname{Hom}(1_G, \delta_A^0) \cap \operatorname{Im}\operatorname{Hom}(\mu_G, 1_{A^1}) \tag{21·1}$$

for each G in Γ.

Proposition 21·1. *A simple extension is Γ-pure if and only if it represents an element of the closed E-functor Φ with the modules F/G ($G \in \Gamma$) as projectives.*

Proof. It suffices to prove the proposition when Γ has exactly one member G. Let A^* be a simple extension. Since F is free the commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathrm{Hom}\,(F/G, A^1) & \to & \mathrm{Hom}\,(F/G, A^2) \\ & & & & \downarrow & & \downarrow \\ 0 \to & \mathrm{Hom}\,(F, A^0) & \overset{\delta_A^0}{\to} & & \mathrm{Hom}\,(F, A^1) & \to & \mathrm{Hom}\,(F, A^2) \to 0 \\ & \downarrow & & & \downarrow & & \downarrow \\ 0 \to & \mathrm{Hom}\,(G, A^0) & \overset{\delta_A^0}{\to} & & \mathrm{Hom}\,(G, A^1) & \to & \mathrm{Hom}\,(G, A^2) \end{array}$$

has exact rows and columns. Element chasing shows that (21·1) is satisfied if and only if the first row is an epimorphism. Since the latter is the condition for A^* to belong to $\tilde{\Phi}$ the result follows.

Let $\aleph$ be the cardinal number of a free basis of F.

First take Γ to be the set of all submodules of F. Then A^* is a Γ-pure extension if and only if each system of linear equations over $\delta_A^0 A^0$ which is solvable in A^1 is also solvable in $\delta_A^0 A^0$. For $\aleph = 1$ this definition coincides with a definition due to Kertesz (1957). For arbitrary $\aleph$ a trivial modification yields Gacsalyi's definition of $\aleph$-pure subgroups of abelian groups. Fuchs (1958) noted the connexion of the latter with E-functors.

Secondly, Cohn (1959) defined $\delta_A^0 A^0$ to be a pure submodule of A^1 if $V \otimes_R A^*$ is exact for each right R-module V. By his theorem 2·4, this coincides with our definition of Γ-pure submodules if, for any infinite cardinal $\aleph$, we take Γ to be the set of finitely generated submodules of F.

Thirdly, if $\aleph = 1$ we can identify F with R and Γ with a set of left ideals of R. If $r \in R$, an Rr-system of linear equations over a module A reduces to one equation $rx = a$, where $a \in A$ and x is 'unknown'. So if Γ contains only principal ideals, a simple extension A^* is Γ-pure if and only if there is a *set* equality $\delta_A^0 A^0 \cap rA^1 = \delta_A^0 rA^0$ for each r such that $Rr \in \Gamma$. Buchsbaum (1959) gives this definition for Γ the set of all principal ideals of a commutative ring R.

Finally let R be a Dedekind domain and Ψ be the E-functor associated in § 20 with a set $\mathfrak{S}$ of non-zero ideals of R. Then Ψ is the E-functor with the modules R/J ($J \in \mathfrak{S}^*$) as projectives. So by proposition 21·1, $\tilde{\Psi}$ is the class of all $\mathfrak{S}^*$-pure extensions of R-modules.

22. E-functors defined by the module structure of Ext

In § 19 we showed that an E-functor Θ takes values in the category of modules over the centre $R_{\mathfrak{C}}$ of the category. Hence each element r of $R_{\mathfrak{C}}$ determines two new E-functors $r\Theta$ and Θ_r with values

$$r\Theta(A,B) = \mathrm{Im}\,[\Theta(A,B) \overset{r}{\to} \Theta(A,B)]$$

and

$$\Theta_r(A,B) = \mathrm{Ker}\,[\Theta(A,B) \overset{r}{\to} \Theta(A,B)].$$

The main theorem of this section is:

Theorem 22·1. *If Θ is closed and M is a non-empty subset of $R_{\mathfrak{C}}$, then*

(i) $\bigcap_{r \in M} r\Theta$ *is a closed E-functor;*

(ii) $\sum_{r \in M} \Theta_r$ *is a closed E-functor if M is closed under multiplication.*

First we establish two lemmas.

LEMMA 22·1. *A necessary and sufficient condition for a simple extension A^* to represent an element $a = ra_0$ of $r\,\mathrm{Ext}^1(A^2, A^0)$ is that there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & A^0 & \overset{\delta^0}{\to} & A^1 & \overset{\delta^1}{\to} & A^2 & \to & 0 \\ & & \| & & \downarrow \lambda_0 & & \downarrow r & & \\ 0 & \to & A^0 & \overset{\delta^0_0}{\to} & A^1_0 & \overset{\delta^1_0}{\to} & A^2 & \to & 0 \\ & & \downarrow r & & \downarrow \lambda & & \| & & \\ 0 & \to & A^0 & \overset{\delta^0}{\to} & A^1 & \overset{\delta^1}{\to} & A^2 & \to & 0 \end{array}$$

in which the middle row represents a_0. In addition λ_0 and λ may be so chosen that

$$\lambda_0\lambda = r, \quad \lambda\lambda_0 = r.$$

Proof. The first sentence of the lemma is simply a restatement of the case $n = 1$ of (19·2). To prove the second consider a diagram of the above form. We show that λ may be replaced by a morphism λ' which preserves commutativity and satisfies the two required conditions.

Since $\lambda\lambda_0\delta^0 = \delta^0 r = r\delta^0$ and coker $\delta^0 = \delta^1$ there is a morphism μ' in Hom (A^2, A^1) such that $\lambda\lambda_0 = r + \mu'\delta^1$. By commutativity $r\delta^1 = \delta^1\lambda\lambda_0$, so $\delta^1\mu'\delta^1 = 0$. Since δ^1 is an epimorphism with kernel δ^0 it follows that $\mu' = \delta^0\mu$ for some μ in Hom (A^2, A^0). Hence we have

$$\lambda\lambda_0 = r + \delta^0\mu\delta^1. \tag{22·1}$$

A similar argument shows that
$$\lambda_0\lambda = r + \delta^0_0\rho\delta^1_0, \tag{22·2}$$

where ρ belongs to Hom (A^2, A^0). By equating $\lambda(\lambda_0\lambda)$ and $(\lambda\lambda_0)\lambda$ one finds easily that

$$\mu = r\rho. \tag{22·3}$$

Now put $\lambda' = \lambda - \delta^0\rho\delta^1_0$. Then it follows from the commutativity of the diagram and (22·1), (22·2), (22·3) that λ' may be substituted in place of λ, and that $\lambda'\lambda_0 = r$, $\lambda_0\lambda' = r$.

LEMMA 22·2. *If A^* is a simple extension, the following statements are equivalent.*

(i) *A^* represents an element of* Ext^1_r.

(ii) *δ^0_A is a right factor of r_{A^0}.*

(iii) *δ^1_A is a left factor of r_{A^2}.*

(iv) *There exist ρ^i in* Hom (A^{i+1}, A^i) $(i = 0, 1)$ *such that $r_{A^1} = \delta^0_A\rho^0 + \rho^1\delta^1_A$, $\rho^0\delta^0_A = r_{A^0}$, and $\delta^1_A\rho^1 = r_{A^2}$.*

Proof. Let A^* represent a in Ext^1. Since $ra = r_{A^0}a = ar_{A^2}$, the equivalence of (i) and (ii) and (i) and (iii) is immediate. Since (iv) implies (ii) it remains to prove that (ii) implies (iv). Let $r_{A^0} = \rho^0\delta^0_A$ for some ρ^0 in Hom (A^1, A^0). Then $\delta^0_A\rho^0\delta^0_A = \delta^0_A r = r\delta^0_A$. So, since coker $\delta^0_A = \delta^1_A$, there exists ρ^1 in Hom (A^2, A^1) such that $r_{A^1} = \delta^0_A\rho^0 + \rho^1\delta^1_A$. Hence $(r_{A^2} - \delta^1_A\rho_1)\,\delta^1_A = 0$ and since δ^1_A is an epimorphism it follows that $r_{A^2} = \delta^1_A\rho^1$. So the lemma is proved.

Now we prove theorem 22·1. Since the intersection of closed E-functors is closed we need only prove (i) when M has one member r. We shall prove that $r\Theta$ is closed on the right; there is a similar proof that it is closed on the left.

Let $a_0 \in \Theta(A^2, A^0)$ and A^* represent $a = ra_0$ in $r\Theta$. Then there is a diagram like that of lemma 22·1 with $\lambda_0\lambda = r$ and $\lambda\lambda_0 = r$. Let $X \in \mathfrak{C}$, and $x = ry \in r\Theta(X, A^1)$, where $y \in \Theta(X, A^1)$, have image 0 under δ^1. Then $0 = \delta^1 ry = r\delta^1 y = \delta^1_0 \lambda_0 y$. Since Θ is closed, there exists z in $\Theta(X, A^0)$ such that $\delta^0 z = \lambda_0 y$. Hence

$$x = ry = \lambda\lambda_0 y = \lambda\delta^0_0 z.$$

So by commutativity $x = \delta^0 rz$, which belongs to $\delta^0 r\Theta(X, A^0)$. This now proves that $r\Theta(X, A^*)$ is exact at $r\Theta(X, A^1)$; that is, that $r\Theta$ is right closed.

In proving (ii) we abbreviate $\sum_{r\in M} \Theta_r$ to Φ. We shall show that the f. class of Φ-monomorphisms satisfy axiom (e_1) for h.f. classes; the proof that the Φ-epimorphisms satisfy axiom (e_2) is quite similar. Let $a \in \Phi$. Then $a = a_1 + \ldots + a_k$ where $r_i a_i = 0$ for certain $r_1, \ldots, r_k$ in M. Since M is multiplicatively closed, there exists r in M such that $ra = 0$. Hence the condition for Θ-monomorphisms α and β to be Φ-monomorphisms is that M contains elements r_α, r_β such that α is a right factor of r_α and β is a right factor of r_β (lemma 22·2 (i), (ii)). If $\beta\alpha$ is defined, it follows (using (19·1)) that it is a right factor of $r_\beta r_\alpha$; moreover, Θ being closed, $\beta\alpha$ is a Θ-monomorphism. Hence $\beta\alpha$ is a $\Theta_{r_\beta r_\alpha}$-monomorphism. Since M is multiplicatively closed $r_\beta r_\alpha \in M$, so we conclude that $\beta\alpha$ is a Φ-monomorphism and the Φ-monomorphisms satisfy axiom (e_1) for h.f. classes.

Proposition 22·1. *For any E-functor Θ and endomorphism r, $r\Theta$ is central in Θ.*

Proof. Each element u of $r\Theta\cdot\Theta$ has the form $u = rx\cdot y$ for x, y in Θ. By (19·2)

$$u = rx\cdot y = xr\cdot y = x\cdot ry \in \Theta\cdot r\Theta,$$

so $r\Theta\cdot\Theta \subset \Theta\cdot r\Theta$. The opposite inclusion may be proved similarly.

Let $\mathrm{Ext}^1_t(A, B)$ and $d\,\mathrm{Ext}^1(A, B)$ denote the maximal torsion subgroup and the maximal divisible subgroup of $\mathrm{Ext}^1(A, B)$. Then Ext^1_t and $d\,\mathrm{Ext}^1$ are E-functors.

Proposition 22·2. Ext^1_t, $\bigcap_{n\geq 1} n\,\mathrm{Ext}^1$, *and* $d\,\mathrm{Ext}^1$ *are closed E-functors.*

Proof. The closure of Ext^1_t and $\bigcap_{n\geq 1} n\,\mathrm{Ext}^1$ follow from theorem 22·1 with $\Theta = \mathrm{Ext}^1$ and M the set of positive integral multiples of the endomorphism 1. We use an idea of Fuchs's to prove that $d\,\mathrm{Ext}^1$ is closed. For each ordinal number v define E-functors $\Theta^{(u)}$ for $0 \leq u \leq v$ by $\Theta^{(0)} = \mathrm{Ext}^1$, $\Theta^{(u+1)} = \bigcap_{n\geq 1} n\Theta^{(u)}$ for $0 \leq u < v$, and $\Theta^{(u)} = \bigcap_{u'<u} \Theta^{(u')}$ for limit ordinals $u \leq v$. By theorem 22·1 (i) and proposition 1·3 each of these E-functors is closed. Now for each pair of objects A, B there exists an ordinal $v(A, B)$ such that $\Theta^{(v)}(A, B) = d\,\mathrm{Ext}^1(A, B)$ for ordinals $v > v(A, B)$. Let $0 \to B \to C \to A \to 0$ represent an element of $d\,\mathrm{Ext}^1(A, B)$ and X be an object of $\mathfrak{C}$. Then exactness of $d\,\mathrm{Ext}^1(X, B) \to d\,\mathrm{Ext}^1(X, C) \to d\,\mathrm{Ext}^1(X, A)$ follows from the closure of $\Theta^{(v)}$ provided we choose $v > \max(v(A, B), v(X, B), v(X, C), v(X, A))$. Hence $d\,\mathrm{Ext}^1$ is closed on the right, and a similar argument shows that it is also closed on the left.

23. Projectives and injectives for Ext^1_r

Let r be an endomorphism of an abelian category $\mathfrak{C}$ and Ext^1_r be the E-functor defined in § 22. We say that an object A is r-regular if r_A is an automorphism.

Proposition 23·1. (i) *An r-regular object is projective and injective for* Ext^1_r.

(ii) *If X is rR-trivial, then* $\mathrm{Ext}^1_r(X, \ ) = \mathrm{Ext}^1(X, \ )$ *and* $\mathrm{Ext}^1_r(\ , X) = \mathrm{Ext}^1(\ , X)$.

Proof. (i) If r_A is an automorphism, so are $\mathrm{Ext}^1(r_A, 1_B)$ and $\mathrm{Ext}^1(1_B, r_A)$ for all objects B in $\mathfrak{C}$. Hence $r\colon \mathrm{Ext}^1(A,B) \to \mathrm{Ext}^1(A,B)$ and $r\colon \mathrm{Ext}^1(B,A) \to \mathrm{Ext}^1(B,A)$ have trivial kernels for all B in $\mathfrak{C}$. So A is Ext^1_r-projective and Ext^1_r-injective.

(ii) X is rR-trivial if and only if $r_X = 0$. If $r_X = 0$, then $r\colon \mathrm{Ext}^1(X,A) \to \mathrm{Ext}^1(X,A)$ and $r\colon \mathrm{Ext}^1(A,X) \to \mathrm{Ext}^1(A,X)$ are zero morphisms for each A in $\mathfrak{C}$.

We recall that a category $\mathfrak{C}$ is hereditary if Ext^2 vanishes. In an hereditary category sub-objects of projectives are projective and quotient objects of injectives are injective.

THEOREM 23·1. *Let $\mathfrak{C}$ be hereditary and have sufficient projectives. An object of $\mathfrak{C}$ is projective for Ext^1_r, where $r \in R_{\mathfrak{C}}$, if and only if it is the direct sum of an r-regular object and a projective.*

Proof. Proposition 23·1 (i) proves the direct implication.

Let P be projective for Ext^1_r and $0 \to X \to Y \overset{\eta}{\to} P \to 0$ be a projective representation of P. For the principal ideal $I = rR$, abbreviate the notation $\kappa_I, \kappa^I, A_I, A^I$ of § 19 to $\kappa_r, \kappa^r, A_r, A^r$. Consider the epimorphism $\kappa^r(P)\,\eta\colon Y \to P^r$. Since P^r is rR-trivial it follows from proposition 23·1 (ii) that $\kappa^r(P)\,\eta$ is an Ext^1_r-epimorphism. But P is Ext^1_r-projective. So there exists a morphism ρ in $\mathrm{Hom}(P, Y)$ such that $\kappa^r(P) = \kappa^r(P)\,\eta\rho$. Write Z for $\mathrm{Im}\,\rho$ and $\lambda\colon L \to P$ for $\ker\rho$. Since $\mathfrak{C}$ is hereditary and Y is projective, Z is projective. Hence λ splits.

To complete the proof it suffices to show that L is r-regular. As in § 19 the morphism λ induces functorially a morphism $\lambda^r\colon L^r \to P^r$ for which $\lambda^r\kappa^r(L) = \kappa^r(P)\,\lambda$. Since $\lambda = \ker\rho$, the definition of ρ shows that $\kappa^r(P)\,\lambda = 0$. So $\lambda^r\kappa^r(L) = 0$. Now λ has a left inverse. Hence λ^r is a monomorphism and $\kappa^r(L) = 0$. This implies that $L^*\colon 0 \to L_r \to L \overset{r_L}{\to} L \to 0$ is an exact sequence. The object L_r is rR-trivial so by proposition 23·1 (ii) $L^* \in \widetilde{\mathrm{Ext}}{}^1_r(L, L_r)$. Since L is a direct summand of P, it is Ext^1_r-projective. So L^* splits and r_L has a right inverse. By (19·1) this is also a left inverse. Thus r_L is an automorphism and L is r-regular.

This proof is adapted from the proof of Folgerung 3·2 in (Baer 1958). The statement of theorem 23·1 remains valid on replacing projective by injective.

We finish this section with an example of a category with sufficient projectives and injectives which has a closed E-functor without sufficient projectives or injectives. In proposition 22·2 we showed that the torsion subgroup E-functor Ext^1_t on any category is closed. Let Z and Q denote the additive groups of integers and rational numbers, and $T = Q/Z$. We prove that in the category of abelian groups Z has no Ext^1_t-injective representation, and T has no Ext^1_t-projective representation.

Since $\mathrm{Ext}^1_t = \bigcup_{n\geqslant 1} \mathrm{Ext}^1_n$, theorem 23·1 shows that Ext^1_t-projectives are torsion free. The injective form of theorem 23·1 shows that Ext^1_t-injectives are divisible. So Z is not Ext^1_t-injective and T is not Ext^1_t-projective. By lemma 22·2 the condition for a monomorphism $\alpha\colon Z \to D$ to be an Ext^1_t-monomorphism is that α is a right factor of $n1_z$ for some positive integer n. This is not possible if D is divisible, for then $\mathrm{Hom}(D, Z) = 0$. So Z cannot have an Ext^1_t-injective representation. Since $nT \neq 0$ for each positive integer n and $\mathrm{Hom}(T, P) = 0$ for torsion free P, a similar argument shows that T cannot have an Ext^1_t-projective representation.

24. HOCHSCHILD E-FUNCTORS

Let $\mathfrak{C}$ be the category of *left* modules over a ring R. Hochschild (1956) associated with each subring of R a right and a left resolution of $\mathfrak{C}$. We shall determine the associated E-functors.

Let R_1 be a subring of R and $\mathfrak{C}_1$ be the category of *left* R_1-modules. We write F for the functor from $\mathfrak{C}$ to $\mathfrak{C}_1$ which converts R-modules to R_1-modules by restriction of operators. Also we consider two functors $G, G': \mathfrak{C}_1 \to \mathfrak{C}$ with values

$$GB = \operatorname{Hom}_1(R, B), \quad G'B = R \otimes_1 B. \tag{24·1}$$

(In §§24 to 28 we write Hom, Hom_1, $\otimes$, $\otimes_1$, etc., for Hom_R, Hom_{R_1}, $\otimes_R$, $\otimes_{R_1}$, etc.).

Let $A \in \mathfrak{C}$ and $B \in \mathfrak{C}_1$. Then well-known associativity formulae yield natural isomorphisms

$$\left.\begin{aligned} \omega&: \operatorname{Hom}(A, GB) \to \operatorname{Hom}_1(FA, B), \\ \omega'&: \operatorname{Hom}(G'B, A) \to \operatorname{Hom}_1(B, FA). \end{aligned}\right\} \tag{24·2}$$

Hence F, G and F, G' are pairs of adjoint functors and determine E-functors Φ and Ψ on $\mathfrak{C}$. Let A^* be a simple extension of R-modules. By proposition 13·3 (ii) A^* belongs to $\tilde{\Phi}$ if and only if $F(\delta_A^0)$ has a left inverse. Similarly A^* belongs to $\tilde{\Psi}$ if and only if $F(\delta_A^1)$ has a right inverse. Since F is an exact functor both conditions are that FA^* splits. So $\Phi = \Psi$, and $\tilde{\Phi}$ may be described as the class of all simple extensions of R-modules which split over R_1. We call Φ the Hochschild E-functor on $\mathfrak{C}$ determined by R_1.

The morphism $\mu_A: A \to GFA$ defined in §13 coincides here with the natural injection of A into $\operatorname{Hom}_1(R, A)$. By proposition 14·1 each R-module A has a Φ-injective representation

$$0 \to A \xrightarrow{\mu_A} \operatorname{Hom}_1(R, A) \to \operatorname{Coker} \mu_A \to 0.$$

Similarly each R-module has a Φ-projective representation

$$0 \to \operatorname{Ker} \epsilon_A \to R \otimes_1 A \xrightarrow{\epsilon_A} A \to 0,$$

where ϵ_A is the natural projection. These representations are precisely the component simple extensions of Hochschild's 'canonical' resolutions (Hochschild 1956, §2).

We shall write $\mathfrak{C}'$, $\mathfrak{C}_1'$ for the category of *right* R-, R_1'-modules. Then we have:

Proposition 24·1. (*a*) *Let R be projective as a member of* $\mathfrak{C}_1$. *Then:* (i) *GF is exact;* (ii) *$G'FP$ is projective in $\mathfrak{C}$ whenever P is projective in* $\mathfrak{C}$.

(*a*′) *Let R be flat as a member of* $\mathfrak{C}_1'$. *Then:* (i) *$G'F$ is exact;* (ii) *GFM is injective in $\mathfrak{C}$ whenever M is injective in* $\mathfrak{C}$.

Proof. (*a*) (i) and (*a*)′ (i) are trivial consequences of (24·1). Let $A^* \in \widetilde{\operatorname{Ext}}^1$ and $P \in \mathfrak{C}$. By (24·2) there are complex isomorphisms

$$\operatorname{Hom}(G'FP, A^*) \xrightarrow{\omega'} \operatorname{Hom}_1(FP, FA^*) \xrightarrow{\omega^{-1}} \operatorname{Hom}(P, GFA^*).$$

Since GF is exact and P is projective, it follows that $G'FP$ is projective.

A similar argument proves (*a*)′ (ii).

Corollary. *Φ is central in* Ext^1 *if R is projective in $\mathfrak{C}_1$ and flat in* $\mathfrak{C}_1'$.

Proof. Propositions 16·3 and 24·1.

Finally, writing Φ' for the Hochschild E-functor on $\mathfrak{C}'$ determined by R_1, we note

Proposition 24·2. *The functor $\otimes$ on $\mathfrak{C}' \times \mathfrak{C}$ is (Φ', Φ)-cobalanced.*

Proof. If $A^* \in \tilde{\Phi}$ and P is Φ'-projective, we must show that $P \otimes A^*$ is exact. Since P is a direct factor of $P \otimes_1 R$ it suffices to prove that $(P \otimes_1 R) \otimes A^*$ is exact. Since

$$(P \otimes_1 R) \otimes A^* \cong P \otimes_1 A^*$$

and A^* splits over R_1, this is obvious. Similarly, one proves that $A^* \otimes P$ is exact whenever $A^* \in \tilde{\Phi}'$ and P is Φ-projective.

25. The Hochschild–Serre spectral sequence

Let R be the integral group ring $Z(G)$ of a group G and R_1 the integral group ring of a subgroup G_1 of G. Following the notation of § 24 we write K_ϕ, K'_ϕ for the classes of Φ-projective, Φ'-projective resolutions of $\mathfrak{C}$, $\mathfrak{C}'$. Also we write K, K' for the class of projective resolutions of $\mathfrak{C}$, $\mathfrak{C}'$.

For $A' \in \mathfrak{C}'$, $A \in \mathfrak{C}$, we denote $A' \otimes A$ by $T(A', A)$. From propositions 12·1, 24·2 there are natural equivalences of functors:

$$K_\phi T \leftarrow (K'_\phi \times K_\phi)\, T \rightarrow K'_\phi T.$$

Next $R = Z(G)$ is free as a left G_1-module and a right G_1-module. So by the corollary to proposition 24·1 both Φ and Φ' are central. Then proposition 12·2 shows that $K_n T\ (\cong \mathrm{Tor}_n^{Z(G)})$ is (Φ', Φ)-cobalanced for all n; and from theorem 12·2 there are canonical exact couple isomorphisms

$$(K_\phi, K)\, T \cong (K'_\phi * K)\, T \cong (K'_\phi, K')\, T \cong (K_\phi * K')\, T. \tag{25·1}$$

We shall show that if G_1 is a normal subgroup of G, then the spectral sequence associated with the exact couple $(K_\phi, K)\, T(A', Z)$—as usual G acts trivially on Z—is isomorphic to the Hochschild–Serre spectral sequence (1953)

$$H_*(G/G_1;\, H_*(G_1;\, A')) \Rightarrow H_*(G;\, A') \tag{25·2}$$

of group homology theory, where $H_*(G;\, A')$ is defined to be $KT(A', Z)$.

Assume that G_1 is normal in G and P_* is a projective resolution of Z in the category of left G/G_1-modules. Let X'_* be a K'-resolution of A'. Cartan & Eilenberg (1956, chap. XVI, §§ 4, 5, 6) prove that one spectral sequence on the double complex

$$M_{**} = X'_* \otimes_{G_1} Z \otimes_{G/G_1} P_*$$

collapses to $H_*(G;\, A')$, and the other is (25·2). Now consider P_* as a complex of G-modules on which G_1 acts trivially. Hochschild (1956, § 6) showed that P_*, so regarded, is a K_ϕ-resolution of Z. Furthermore M_{**} is canonically isomorphic to $X'_* \otimes P_* = T(X'_*, P_*)$. So its non-collapsing exact couple is isomorphic to $(K_\phi * K')\, T(A', Z)$. Our assertion now follows from (25·1).

The groups $K_\phi T(A', Z)$ of positive degree coincide with the positive degree homology groups $H_*(G, G_1;\, A')$ of G (mod G_1), with coefficients in A', of Adamson (1954) and Hochschild (1956). If $G_1 = (1)$, then $H_*(G, G_1;\, A')$ coincides with $H_*(G;\, A')$. It has been proved by Adamson (1954, theorem 3·2) that when G_1 has a subgroup G_2 normal in G,

$$H_*(G, G_1;\, A') \cong H_*(G/G_2, G_1/G_2;\, A' \otimes_G Z(G/G_2)).$$

We mention that this result follows from the observation that a resolution of Z by G/G_2-modules which splits as a G_1/G_2-complex, and whose components are projective over G/G_2-epimorphisms with G_1/G_2-inverses, is a K_ϕ-resolution of Z when its components are regarded as G-modules.

Similar results may be obtained for Hom and the Hochschild–Serre sequence for the cohomology of a group.

26. Pairs of Hochschild E-functors

Let R be a ring and $\mathfrak{C}$ be the category of *left* R-modules. Write Φ and Θ for the Hochschild E-functors on $\mathfrak{C}$ determined by a subring R_1 of R and a subring R_2 of R_1. We shall obtain sufficient conditions for Φ to be central in Θ.

Write $\mathfrak{D}(\mathfrak{D}')$ for the category of *left* R- and *right* R_2- (*right* R- and *left* R_2-) bimodules. We shall have to consider the natural $\mathfrak{D}$-epimorphism

$$\epsilon\colon\ R\otimes_2(R\otimes_1 R)\rightarrow R\otimes_1 R \tag{26·1}$$

which maps $r\otimes_2 r'\otimes_1 r''$ to $rr'\otimes_1 r''$, and the $\mathfrak{D}'$-epimorphism

$$\epsilon'\colon\ (R\otimes_1 R)\otimes_2 R\rightarrow R\otimes_1 R \tag{26·1'}$$

which maps $r\otimes_1 r'\otimes_2 r''$ to $r\otimes_1 r'r''$. With F, G, G' as in § 24 we have:

PROPOSITION 26·1. (*a*) *If* ϵ *has a right* $\mathfrak{D}$*-inverse, then:* (i) $G'FP$ *is* Θ*-projective if* P *is* Θ*-projective*; (ii) $GFA^*\in\tilde{\Theta}$ *if* $A^*\in\tilde{\Theta}$.

(*b*) *If* ϵ' *has a right* $\mathfrak{D}'$*-inverse, then:* (i) GFQ *is* Θ*-injective if* Q *is* Θ*-injective;* (ii) $G'FA^*\in\tilde{\Theta}$ *if* $A^*\in\tilde{\Theta}$.

Proof. We prove only (*a*). The proof of (*b*) is similar. Since there are sufficient Θ-projectives of the form $R\otimes_2 B$ ($B\in\mathfrak{C}_2$), we can assume that $P=R\otimes_2 B$. Let μ be a right $\mathfrak{D}$-inverse of ϵ. Then $\mu\otimes_2 1_B$ is a right $\mathfrak{C}$-inverse of $\epsilon\otimes_2 1_B$. Thus $R\otimes_1 R\otimes_2 B$ (i.e. $G'F(R\otimes_2 B)$) is a direct summand in $\mathfrak{C}$ of $R\otimes_2(R\otimes_1 R\otimes_2 B)$. The latter is Θ-projective. So $G'F(R\otimes_2 B)$ is Θ-projective, and (i) is proved.

We now deduce (ii). Let $B\in\mathfrak{C}_2$, and $A^*\in\tilde{\Theta}$. By (24·1) and standard associativity formulae, there exist complex isomorphisms

$$\begin{aligned}\mathrm{Hom}_2(B,GFA^*)\cong\mathrm{Hom}_1(R\otimes_2 B,\,FA^*)\cong\mathrm{Hom}_1(R\otimes_2 B,\,\mathrm{Hom}\,(R,A^*))\\ \cong\mathrm{Hom}\,(G'F(R\otimes_2 B),\,A^*).\end{aligned}$$

Since $A^*\in\tilde{\Theta}$ and $G'F(R\otimes_2 B)$ is Θ-projective, the last sequence is exact. So the first sequence is exact. Hence, since B may be any R_2-module, GFA^* splits as a sequence of R_2-modules. So $GFA^*\in\tilde{\Theta}$, and (ii) is proved.

From proposition 16·3 we deduce:

COROLLARY. *If* ϵ *has a right* $\mathfrak{D}$*-inverse and* ϵ' *has a right* $\mathfrak{D}'$*-inverse, then* Φ *is central in* Θ.

Let R, R_1, R_2 be the integral group rings of a group G, a subgroup G_1 of G, and a subgroup G_2 of G_1. We shall construct a right $\mathfrak{D}$-inverse of ϵ. There is a similar construction of a right $\mathfrak{D}'$-inverse of ϵ'.

Let $G=\bigcup_{i\in I}G_1y_iG_2$ be a double coset decomposition of G (mod G_1,G_2). For $i\in I$, let

$$G_2=\bigcup_{j\in J_i}(G_2\cap y_i^{-1}G_1y_i)\,z_{ij}$$

be a left coset decomposition of G_2 (mod $G_2\cap y_i^{-1}G_1y_i$). Then each element of G has exactly one representation in the form

$$x_1y_iz_{ij},\quad \text{where}\quad x_1\in G_1,\ i\in I,\ j\in J_i.$$

Now $R=Z(G)$ is a free left G_1-module on a set of left coset representatives of G (mod G_1). Hence $R\otimes_1 R$ is a free abelian group with the elements

$$x\otimes_1 y_iz_{ij}\quad (x\in G,\ i\in I,\ j\in J_i)$$

as a free basis. The abelian group homomorphism

$$\mu\colon\ R\otimes_1 R\rightarrow R\otimes_2 R\otimes_1 R$$

defined by
$$\mu(x\otimes_1 y_iz_{ij})=xy_i\otimes_2 y_i^{-1}\otimes_1 y_iz_{ij}$$

is a right inverse of ϵ. Also it is clearly a left R-module homomorphism, so to show that it is a right $\mathfrak{D}$-inverse it suffices to verify that for u in G_2

$$\mu(x\otimes_1 y_i z_{ij} u) = \mu(x\otimes_1 y_i z_{ij})\, u.$$

To do this, write $z_{ij}u = vz_{ik}$, where $k \in J_i$ and $v \in G_2 \cap y_i^{-1}G_1 y_i$. Let w be the element of G_1 for which $v = y_i^{-1}wy_i$. Then we have

$$\mu(x\otimes_1 y_i z_{ij} u) = \mu(xw\otimes_1 y_i z_{ik}) = xwy_i\otimes_2 y_i^{-1}\otimes_1 y_i z_{ik}$$

and it is easy to verify that the last expression is $\mu(x\otimes_1 y_i z_{ij})\, u$.

It is a consequence of this result that the isomorphisms of exact couples in (25·1) remain valid when K is replaced by K_θ.

27. On E-functors with exponents

An *exponent* of an E-functor Θ on a category $\mathfrak{C}$ is defined to be an endomorphism $r \neq 0$ of $\mathfrak{C}$ such that $r\Theta = 0$. In this section we assume that Θ has sufficient injectives, so that the Θ-injective resolutions form a right resolution K_θ of $\mathfrak{C}$.

Proposition 27·1. *Let r be an exponent of Θ. Then X^* in K_θ admits a family $\{\sigma^n\}_{n\geqslant 1}$ of morphisms $\sigma^n\colon X^n \to X^{n-1}$ such that*

$$\sigma^{n+1}\delta^n_X + \delta^{n-1}_X\sigma^n = r \quad \text{for} \quad n \geqslant 1.$$

Proof. As in §3 write $\delta^n_X = \mu^{n+1}_X\eta^{n+1}_X$ where $\mu^n_X\colon \bar{X}^n \to X^n$ is the kernel of δ^n_X. In this proof we shall omit the suffix 'X'. The simple extension

$$0 \to \bar{X}^n \xrightarrow{\mu^n} X^n \xrightarrow{\eta^{n+1}} \bar{X}^{n+1} \to 0$$

belongs to $\tilde{\Theta}$. Since $r\Theta = 0$ it follows from lemma 22·2 that there are morphisms $\pi^n\colon X^n \to \bar{X}^n$ and $\nu^n\colon \bar{X}^{n+1} \to X^n$ such that

$$\pi^n\mu^n = r, \quad \eta^{n+1}\nu^n = r \quad \text{and} \quad \mu^n\pi^n + \nu^n\eta^{n+1} = r. \tag{27·1}$$

We determine the σ^n inductively, together with morphisms $\tau^n\colon \bar{X}^{n+1} \to \bar{X}^n$ related to the σ^n by the formulae

$$\sigma^n\mu^n = \nu^{n-1} + \mu^{n-1}\tau^{n-1} \quad \text{for} \quad n \geqslant 1, \tag{27·2}$$

$$\eta^n\sigma^n = \pi^n - \tau^n\eta^{n+1} \quad \text{for} \quad n \geqslant 1. \tag{27·3}$$

Let $\tau^0 = 0$. Since X^0 is Θ-injective and μ^1 is a Θ-monomorphism, ν^0 extends to a morphism $\sigma^1\colon X^1 \to X^0$ satisfying (27·2) for $n = 1$. From (27·1) we obtain $\pi^1\mu^1 = r = \eta^1\nu^0$, and from (27·2) with $n = 1$, $\eta^1\nu^0 = \eta^1\sigma^1\mu^1$. Hence $(\pi^1 - \eta^1\sigma^1)\,\mu^1 = 0$. Since $\operatorname{coker}\mu^1 = \eta^2$, we now choose τ^1 to satisfy (27·3) with $n = 1$.

Now let n be greater than 1 and suppose that σ^m, τ^m $(m = 0, 1, \ldots, n-1)$ satisfy (27·2) and (27·3). Since $\nu^{n-1} + \mu^{n-1}\tau^{n-1}$ belongs to $\operatorname{Hom}(\bar{X}^n, X^{n-1})$, X^{n-1} is Θ-injective, and μ^n is a Θ-monomorphism, there exists $\sigma^n\colon X^n \to X^{n+1}$ satisfying (27·2). As for τ^1 it follows from (27·1), (27·2), and $\operatorname{coker}\mu^n = \eta^{n+1}$, that there is a morphism τ^n satisfying (27·3). So the existence of σ^n, τ^n satisfying (27·2), (27·3) may be proved by induction.

Finally we obtain the formula $\sigma^{n+1}\delta^n + \delta^{n-1}\sigma^n = r$, for $n \geqslant 1$, by eliminating ν^n, π^n from (27·2), (27·3), and the third formula of (27·1).

Let $r^n = r_{X^n}$ for $n \geq 0$. Since r is an endomorphism of $\mathfrak{C}$, r^* is a complex morphism covering $r_{\overline{X}^0}$. Let $T: \mathfrak{C} \to \mathfrak{D}$ be a covariant functor. Then it follows from the proposition that $\hat{K}^n_\theta T(r_{\overline{X}^0}) = 0$, $\check{K}_\theta T(r_{\overline{X}^0}) = 0$, $K^n_\theta T(r_{\overline{X}^0}) = 0$ for all $n > 0$. We state this result as:

THEOREM 27·1. *If Θ has sufficient injectives and an exponent r and if T is an additive functor on $\mathfrak{C}$, then in positive degrees*

$$\hat{K}_\theta T(r) = 0, \quad \check{K}_\theta T(r) = 0, \quad K_\theta T(r) = 0,$$

where $K_\theta T(r)$ is the natural transformation of $K_\theta T$ into itself with values $K_\theta T(r_A)$, etc.

For the remainder of this section we assume that Θ has sufficient injectives and an exponent r, and that s, t are a pair of endomorphisms of $\mathfrak{C}$ such that

$$r \text{ divides } st \quad \text{and} \quad s+t = 1.$$

So s, t behave as orthogonal idempotents, modulo r; and Θ decomposes into the 'direct sum' of E-functors Φ, Ψ (with exponents t and s) defined by

$$\Phi = s\Theta = \Theta \cap \mathrm{Ext}^1_t \quad \text{and} \quad \Psi = t\Theta = \Theta \cap \mathrm{Ext}^1_s. \tag{27·4}$$

For later reference note that

$$sx = x \text{ for } x \in \Phi \quad \text{and} \quad ty = y \text{ for } y \in \Psi. \tag{27·5}$$

Because of the symmetry in the definitions of Φ and Ψ we can interchange their roles in the following results if at the same time we interchange s and t.

We show that Φ and Ψ have sufficient injectives, and the positive degree components of $K_\theta T$, $\check{K}_\theta T$ and $\hat{K}_\theta T$ 'inherit' the decomposition $\Phi \oplus \Psi$ of Θ.

PROPOSITION 27·2. *Let X^* be a K_θ-resolution of an object A. Then there exist a Φ-injective resolution Y^* of A, and a complex morphism $\bar{s}^*: X^* \to Y^*$ covering s_A.*

Proof. We use the notation of proposition 27·1 with $\overline{X}^0 = A$, and write x^n for the image in Θ of $0 \to \overline{X}^n \to X^n \to \overline{X}^{n+1} \to 0$. Let $y^n = sx^n$. We shall prove that any simple extension

$$0 \to \overline{X}^n \to Y^n \to \overline{X}^{n+1} \to 0$$

with image y^n in Θ is a Φ-injective representation of $\overline{X}^n$. The required resolution Y^* is obtained by splicing them together.

By (27·4) $y^n \in \Phi$. So it suffices to prove that Y^n is Φ-injective. Let $B \in \mathfrak{C}$. Since $y^n = sx^n$, lemma 22·1 shows that there is a factorization of s_{Y^n} through X^n. So there is a factorization of $\Phi(1_B, s_{Y^n})$ through $\Phi(B, X^n)$. But the latter vanishes, since $\Phi \subset \Theta$ and X^n is a Θ-injective. So $\Phi(1_B, s_{Y^n})$ vanishes. By (27·5) this is the identity on $\Phi(B, Y^n)$. So $\Phi(B, Y^n)$ vanishes. Since B is arbitrary, Y^n is Φ-injective.

Finally we construct $\bar{s}^*$. Since $sy^n = y^n$ and $y^n = sx^n$, it follows that $sx^n = y^n s$. So there exists a morphism $\bar{s}^n$ of X^n into Y^n making the diagram

$$\begin{array}{ccccccccc} 0 & \to & \overline{X}^n & \to & X^n & \to & \overline{X}^{n+1} & \to & 0 \\ & & s\downarrow & & \bar{s}^n\downarrow & & s\downarrow & & \\ 0 & \to & \overline{X}^n & \to & Y^n & \to & \overline{X}^{n+1} & \to & 0 \end{array}$$

commutative. The morphisms $\bar{s}^0$, $\bar{s}^1$, ... are the components of the required complex morphism $\bar{s}^*$.

In particular this proposition shows that the Φ-injective resolutions form a resolution K_ϕ of the category. Let T be a covariant functor on $\mathfrak{C}$. Since $\Phi \subset \Theta$ there is a τ-transformation

$$\tau^n_\phi\colon K^n_\phi T \to K^n_\theta T,$$

and there are similar transformations for the satellites and cosatellites.

THEOREM 27·2. *For $n > 0$ there exist natural transformations $\pi^n_\phi : K^n_\theta T \to K^n_\phi T$ such that*

$$(a)\ \ \tau^n_\phi \pi^n_\phi = K^n_\theta T(s); \quad (b)\ \ \pi^n_\phi \tau^n_\phi = K^n_\phi T(1). \qquad (27\cdot 6_\phi)$$

Proof. Let X^* be a K_θ-resolution of an object of $\mathfrak{C}$. By proposition 27·2 there exist a Φ-injective resolution Y^* of A and a complex morphism $\bar{s}^*\colon X^* \to Y^*$ covering s_A. Let $\pi^n_\phi(A)$ be the morphism of $K^n_\theta TA$ into $K^n_\phi TA$ induced by $\bar{s}^*$. Since $K_\phi \prec K_\theta$ there exists a complex morphism α^* of Y^* into X^* covering 1_A, and it induces the value $\tau^n_\phi(A)$ of τ^n_ϕ on A. The morphisms $\alpha^*\bar{s}^*$ and $\bar{s}^*\alpha^*$ both cover s_A, so

$$\tau^n_\phi(A)\,\pi^n_\phi(A) = K^n_\theta T(s_A)$$

and

$$\pi^n_\phi(A)\,\tau^n_\phi(A) = K^n_\phi T(s_A).$$

As n is positive, theorem 27·1 shows that $K^n_\phi T(t_A) = 0$. Since $s+t = 1$ the second formula becomes

$$\pi^n_\phi(A)\,\tau^n_\phi(A) = K^n_\phi T(1_A).$$

It follows that $\tau^n_\phi(A)$ is a monomorphism and $\pi^n_\phi(A)$ is an epimorphism. Hence the factorization of $K^n_\theta T(s_A)$ shows that $\pi^n_\phi(A) \cong \operatorname{coim} K^n_\theta T(s_A)$. Thus $\pi^n_\phi(A)$ is the value of a natural transformation π^n_ϕ, and π^n_ϕ is determined by $(27\cdot 6_\phi)$ as $\operatorname{coim} K^n_\theta T(s)$.

THEOREM 27·3. *For $n > 0$ there is a natural decomposition*

$$K^n_\theta T \cong K^n_\phi T \oplus K^n_\psi T,$$

and τ^n_ϕ induces an isomorphism $K^n_\phi T \cong \operatorname{Im} K^n_\theta T(s)$.

Proof. From $(27\cdot 6_\phi)$ (a) and $(27\cdot 6_\psi)$ (a)

$$\tau^n_\phi \pi^n_\phi + \tau^n_\psi \pi^n_\psi = K^n_\theta T(s) + K^n_\theta T(t) = K^n_\theta T(1), \qquad (27\cdot 7)$$

and with $(27\cdot 6_\phi)$ (b) and $(27\cdot 6_\psi)$ (b) this proves the first statement. Since π^n_ϕ is an epimorphism, the second statement follows from $(27\cdot 6_\phi)$ (a).

By similar methods we can obtain decompositions

$$\theta^n T \cong \phi^n T \oplus \psi^n T \quad \text{and} \quad \hat{K}^n_\theta T \cong \hat{K}^n_\phi T \oplus \hat{K}^n_\psi T \quad \text{for} \quad n > 0.$$

Since $K^n_\theta T \cong \theta^n K^0_\theta T$, the first of these shows that

$$K^n_\theta T \cong \theta^n K^0_\theta T \cong \phi^n K^0_\theta T \oplus \psi^n K^0_\theta T.$$

With theorem 27·3 this shows that there are isomorphisms

$$\phi^n K^0_\phi T \cong \phi^n K^0_\theta T \quad \text{and} \quad \psi^n K^0_\psi T \cong \psi^n K^0_\theta T \quad \text{for} \quad n > 0. \qquad (27\cdot 8)$$

We can obtain a similar result for the cosatellites.

We conclude this section by determining some of the terms in the exact couple $(K_\phi, K_\theta)\,T$. Let p and q be positive integers. By (7·9) and (7·10), $K^p_\theta K^q_\theta T = 0$. So theorem 27·3 shows that $K^p_\phi K^q_\theta T = 0$, and with theorem 5·1 this yields

$$E^{pq}_2(K_\phi, K_\theta)\,T = 0.$$

Again theorem 27·3 shows that

$$\phi^p K^q_\theta T \cong \phi^p K^q_\phi T \oplus \phi^p K^q_\psi T.$$

The first term on the right-hand side is isomorphic to $K^{p+q}_\phi T$. Since Φ has exponent t and Ψ has exponent s, the second term, by theorem 27·1, has $\phi^p K^q_\psi T(s)$ and $\phi^p K^q_\psi T(t)$ as exponents. Since $s+t=1$, it vanishes. So from theorem 5·2

$$C^{pq}_2(K_\phi, K_\theta)\, T \cong K^{p+q}_\phi T.$$

From proposition 11·2 with $k=0$ the component of $K^n_\theta T$ with filtration p is the image of $\phi^p K^{n-p}_\theta T$ in $K^n_\theta T$ induced by a τ-transformation. When $n>p$, $\phi^p K^{n-p}_\theta T$ is $K^n_\phi T$ by what has just been proved, and $\phi^n K^0_\theta T$ is $K^n_\phi T$ by (27·8). So the component of $K^n_\theta T$ with filtration p is the image of $K^n_\phi T$ under τ^n_ϕ.

28. Subgroups of finite index

Let G_1 be a subgroup of a group G, G_2 a subgroup of G_1, R, R_1, R_2 the integral group rings of G, G_1, G_2, and Φ, Θ the Hochschild E-functors on the category $\mathfrak{C}$ of left G-modules determined by R_1, R_2. In § 26 we showed that Φ is central in Θ. Throughout this section we assume that G_2 has finite index h in G_1 and G_1 has finite index k in G.

Let $x_1, \ldots, x_h$ be a set of right coset representatives of G_1 (mod G_2). For G-modules A and B, Eckmann (1953) defined the 'transfer' homomorphism

$$\tau\colon \operatorname{Hom}_2(A,B) \to \operatorname{Hom}_1(A,B)$$

by the formula $$\tau(\alpha)\colon a \to \sum_{i=1}^{h} x_i \alpha(x_i^{-1} a) \quad (a \in A,\ \alpha \in \operatorname{Hom}_2(A,B)).$$

The transfer is natural and independent of the choice of coset representatives. In addition

$$\left.\begin{aligned} \tau(\alpha\beta) &= \alpha\tau(\beta) \quad \text{for} \quad \alpha \in \operatorname{Hom}_1(A,B),\ \beta \in \operatorname{Hom}_2(C,A);\\ \tau(\alpha\beta) &= \tau(\alpha)\beta \quad \text{for} \quad \alpha \in \operatorname{Hom}_2(A,B),\ \beta \in \operatorname{Hom}_1(C,A).\end{aligned}\right\} \tag{28·1}$$

Proposition 28·1. $\Phi \supset h\Theta$.

Proof. Let $a \in \Theta$ and A^* be a simple extension of G-modules with image a. The condition for $A^* \in \tilde{\Theta}$ is that it splits over G_2. So δ^1_A has a right inverse α in $\operatorname{Hom}_2(A^2, A^1)$. Using (28·1) we deduce that

$$\delta^1_A \tau(\alpha) = \tau(\delta^1_A \alpha) = \tau(1_{A^2}) = h 1_{A^2}.$$

Hence δ^1_A, in the category of G_1-modules, is a left factor of the endomorphism h. It follows from lemma 22·2 that a simple extension of G-modules with image ha in Θ must split over G_1. Hence $ha \in \Phi$, as required.

If $G_1 = G$, then h is the index $[G\colon G_2]$ of G_2 in G and $\Phi = 0$. Hence we have

Corollary. *Θ has an exponent* $[G\colon G_2]$.

Theorem 27·1 now shows, in particular, that the relative homology groups $H_n(G, G_2; A')$ have exponent $[G\colon G_2]$ for all $n>0$.

We now assume that h and k are coprime integers. Let s and t be multiples of h and k such that $s+t=1$. Since Θ has an exponent $[G\colon G_2] = hk$ and Φ has an exponent $[G\colon G_1] = k$, st is an exponent of Θ and t is an exponent of Φ. With proposition 28·1 this shows that $s\Theta \supset \Phi \supset h\Theta$, and hence $s\Theta = \Phi = h\Theta$. So we can apply the theory of the preceding section to Φ. Suppose that T is a functor with values in a category of abelian groups. Then $K^n_\theta\, T(s_A)$ is multiplication of the abelian group $K^n_\theta\, T(A)$ by the integer s. So $\operatorname{Im} K^n_\theta\, T(s_A) = sK^n_\theta\, T(A)$. From theorem 27·1 $K^n_\theta\, T(A)$ has exponent hk when $n \geqslant 1$. Hence $\operatorname{Im} K^n_\theta\, T(s_A) = hK^n_\theta\, T(A)$. So it follows from theorem 27·3 that τ^n_ϕ induces natural isomorphisms

$$K^n_\phi\, T(A) \cong hK^n_\theta\, T(A) \quad (n \geqslant 1).$$

Since h and k are coprime the right side is the subgroup of elements in $K^n_\theta\, T(A)$ with orders which divide k.

We apply this to relate the homology groups $H_n(G, G_1; A')$ and $H_n(G, G_2; A')$. These groups are computed from projective resolutions of G-modules so we must interchange τ and π in $(27{\cdot}6_\phi)$ and $(27{\cdot}7)$. The result is:

Theorem 28·1. *If* $[G\colon G_1]$ *and* $[G_1\colon G_2]$ *are coprime, the* τ*-transformation*

$$H_n(G, G_2; A') \to H_n(G, G_1; A') \quad (n \geqslant 1)$$

maps the subgroup of elements of $H_n(G, G_2; A')$ *with orders which divide* $[G\colon G_1]$ *isomorphically onto* $H_n(G, G_1; A')$.

When G_2 is the trivial subgroup, $H_n(G, G_2; A') = H_n(G; A')$. Hence:

Corollary. If G_1 is a subgroup of a finite group G with index coprime to its order, then $H_n(G, G_1; A')$ is isomorphic to the subgroup of elements of $H_n(G; A')$ with orders dividing $[G\colon G_1]$ for all $n \geqslant 1$.

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